On Robust Estimation of Seasonal Autocorrelation Function for Periodic Autoregressive Models

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To my parents...................

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TABLE OF CONTENTS

CHAPTER SIX: CONCLUSIONS AND FUTURE WORK

LIST OF TABLES

Table Number **Page**

Table Number		Page

LIST OF FIGURES

Figure Number		Page

Figure Number		Page

ABSTRACT

AL-Quraan, Areen Mahmoud Mufleh. On Robust Estimation of Seasonal Autocorrelation Function for Periodic Autoregressive Models. Master of Science Thesis, Department of Statistics, Yarmouk University, 2010. (Supervisor: Dr. Abdullah A. Smadi), (Co-Supervisor: Dr. Mohammed Y. Al-Rawwash).

The importance of periodic autoregressive models as plausible models for seasonal time series have recently increased. In this thesis the seasonal autocorrelation function (SACF) of such models is considered.

The main objective of this research is to investigate robust estimation of the SACF for the periodic autoregressive models with different orders and time lags. In this thesis we have considered three estimators for SACF including the ordinary moment estimator beside two new estimators. We have studied via simulation technique (Monte-Carlo simulation) the robustness of these estimators for the presence of additive outliers in the time series in view of bias and MSE.

We found that the moment estimator is highly affected by the existence of additive outliers while the other two estimators seem to be more robust on the basis of bias and MSE.

A real application on a quarterly river–flow time series is carried out. The results of this application assured the simulation results.

Key Words: Periodic autoregressive model, Seasonal autocorrelation function, Robust estimation, Additive outlier, Monte-Carlo simulation.

الملخص

القرعان، عرين محمود مفلح. حول التقديرات المتينة لاقتران الارتباط الذاتي لنماذج الانحدار الذاتي الدورية. رسالة ماجستير في العلوم، قسم الإحصاء، جامعة اليرموك. 2010 (المشرف: الدكتور عبدالله أحمد الصمادي)، (المشرف المساعد: الدكتور محمد يوسف الرواش).

إن أهمية نماذج الانحدار الذاتي الدورية كنماذج محتملة للسلاسل الزمنية الموسمية قد تزايدت بشكل ملحوظ مؤخرا. في هذه الرسالة سوف يتم دراسة اقتران الارتباط الذاتي الدوري لهذه النماذج.

إن الهدف الرئيسي من هذا البحث هو محاولة ايجاد تقديرات متينة لاقتران الارتباط الذاتي الدوري لنماذج الانحدار الذاتي الدورية لمختلف الرتب و مسافات زمنية متعددة. في هذه الرسالة أخذنا بعين الإعتبار ثلاثة تقديرات لاقتران الارتباط الذاتي الدوري تتضمن تقدير العزوم التقليدي بالإضافة إلى تقديرين جديدين. بإستخدام أسلوب المحاكاة (محاكاة مونتي كارلو) درسنا متانة التقديرات الثلاثة بوجود قيم شاذة مضافة إلى السلاسل الزمنية الموسمية من ناحية التحيز و متوسط مربع الأخطاء.

لقد وجدنا أن تقدير العزوم التقليدي قد تأثر بشكل كبير بوجود القيم الشاذة المضافة بينما ظهر أن التقديرين الآخرين كانا أكثر متانة من ناحية التحيز و متوسط مربع الأخطاء.

لقد تم تنفيذ التطبيق على سلسلة زمنية حقيقية تتعلق بمعدلات تدفق الأنهار الربعية. وقد كانت نتائج هذا التطبيق موافقة لنتائجنا المحاكاة السابقة.

الكلمات المفتاحية: نماذج الانحدار الذاتي الدورية، اقتران الارتباط الذاتي، التقديرات المتينة، القيم الشاذة المضافة، محاكاة مونتي كارلو.

LIST OF SYMBOLS

ACF	Autocorrelation function
ACVF	Autocovariance function
AO	Additive outliers
AR	Autoregressive
ARIMA	Autoregressive integrated moving average
ARMA	Autoregressive moving average
IO	Innovative outliers
MA	Moving average
PACF	Partial autocorrelation function
PAR	Periodic Autoregressive
PARMA	Periodic Autoregressive moving average
PMA	Periodic moving average
SACF	Seasonal autocorrelation function
SACVF	Seasonal autocovariance function
SPACF	Seasonal partial autocorrelation function
$\{a_t\}$	White noise process

CHAPTER 1

Introduction

1.1 Preliminaries

A time series can be defined as sequence of observations taken sequentially in time. Time series can be observed in different fields; for example, in agriculture, business, economics, engineering and medical studies. The list of areas in which time series is observed, studied and analyzed is endless. The purpose of time series analysis is generally, to understand or model the stochastic mechanism that gives rise to an observed series, to predict or forecast the future values of a series based on the history of that series and the optimal control of a system (Cryer and Chan, 2008).

A unique feature of time series and their models is that we usually can not assume that the observation arise independently from a common population (or from populations with different means, for examples), that means, the observations are dependent and the order of the observations is, therefore, important (Wei,1990).

1.2 ARIMA Models

The autoregressive integrated moving average (ARIMA) models are considered one of the most popular models used in time series analysis. The seasonal ARIMA model with orders (p, d, q) × (P, D, Q)$_\omega$ is the most general form of ARIMA models written as:

$$\Phi_P(B^\omega)\phi_p(B)(1-B)^d(1-B^\omega)^D X_t = \theta_0 + \Theta_Q(B^\omega)\theta_q(B)a_t, \qquad (1.1)$$

where, ω is the period, $\Phi_P(B^\omega), \phi_p(B\ ), \Theta_Q(B^\omega)$ and $\theta_q(B)$ are the seasonal autoregressive (AR), ordinary AR, seasonal moving average (MA) and ordinary MA operands, respectively, θ_0 is a constant, d and D are the ordinary and seasonal differencing orders respectively, and $\{a_t\}$ is a sequence of independent and identically distributed random variables with mean zero and variance σ_a^2 (Box et al ., 1994).

If D=d=0 and $\omega = 1$, then equation (1.1) is reduced to the ordinary ARMA model with orders (p,q), denoted by ARMA(p,q) and written as:

$$X_t = \theta_0 + \phi_1 X_{t-1} + \phi_2 X_{t-2} + \ldots + \phi_p X_{t-p} + a_t - \theta_1 a_{t-1} - \theta_2 a_{t-2} - \ldots - \theta_q a_{t-q}.$$

The pure AR(p) and pure MA(q) models are special cases of the ARMA(p,q) model when assuming q = 0 and p = 0 respectively. For instance, the AR(1) model can be written as $X_t = \theta_0 + \phi_1 X_{t-1} + a_t$.

In general, ARMA(p,q) model is used for modeling stationary time series. In fact, there are two types of stationarity, namely strict and weak stationarity, but in practice we usually deal with the later type.

Let $\{X_t\}$ be a stochastic process with mean μ_t and autocovariance function $\gamma(t_1,t_2) = Cov(X_{t_1}, X_{t_2})$. Then $\{X_t\}$ is said to be weakly (covariance) stationary if μ_t is constant and the autocovariance function depends on time lag only. Thus, for stationary stochastic process, the mean, variance and the autocovariance function are denoted by μ , γ_0 and γ_k respectively where $\gamma_k = Cov(X_t, X_{t-k})$; $k = 0, \pm 1, \pm 2, \ldots$ (Cryer and Chan, 2008).

For ARMA(p,q) processes, a sufficient condition for stationarity is that all roots of $\phi_p(B) = 0$ should lie outside the unit circle. For example, the condition for AR(1) reduces to $|\phi| < 1$. Also, as a special case of ARMA(p,q) process, the pure MA process are always stationary (Wei,1990).

A very important example of a stationary process is the so-called white noise process (WN), which is defined as a sequence of independent and identically distributed random variables $\{a_t\}$ with zero mean, constant variance σ_a^2. Although the process rarely occurs in applied time series, it plays an important role as a basic building block in the construction of time series models (Wei, 1990).

Finally, for stationary stochastic processes the autocorrelation function (ACF) denoted by ρ_k is:

$$\rho_k = \frac{\gamma_k}{\gamma_0} \ ; \ |k| = 0,1,\ldots .$$

In the next chapter, we will extensively study this function for ARMA processes.

1.3 PARMA Models

The periodic ARMA (PARMA) model is an extension of the ordinary ARMA model that is suitable for modeling seasonal time series. This model consists of ω equations, where ω refers to the period.

Writing t as $(j\omega+\nu)$ then the PARMA$_\omega$ (p(ν), q(ν)) model is written as :

$$\left(1-\phi_1(\nu)B-\ldots-\phi_{p(\nu)}(\nu)B^{p(\nu)}\right)\left(X_{j\omega+\nu}-\mu_{j\omega+\nu}\right)$$
$$=\left(1-\theta_1(\nu)B-\ldots-\theta_{q(\nu)}(\nu)B^{q(\nu)}\right)a_{j\omega+\nu}, \qquad (1.2)$$

where $\nu=1,2....\omega$ denotes the season, j=0,1 ... stands for the year, μ_ν is the mean of season ν, $\{a_{j\omega+\nu}\}$ is a periodic white noise process with zero mean and periodic variances σ_ν^2; p(ν) and q(ν) are the AR and MA orders of season ν respectively.

The constant order models PARMA$_\omega$(p,q), PAR$_\omega$(p), PMA$_\omega$(q) as well as the ordinary ARMA (p ,q) models are special cases of the model in (1.2). As for the AR(1) model in context of ARMA models, the PAR$_\omega$(1) could be the most important model among PARMA models. For instance, the zero-mean PAR$_\omega$(1) model is given by:

$$X_{j\omega+\nu} = \phi_1(\nu) X_{j\omega+\nu-1} + a_{j\omega+\nu}.$$

For example, taking ω=4, then the zero-mean PAR$_4$(1) model is written explicitly as:

$$\left.\begin{aligned} X_{j4+1} &= \phi_1(1) X_{(j-1)4+4} + a_{j4+1} \\ X_{j4+2} &= \phi_1(2) X_{j4+1} + a_{j4+2} \\ X_{j4+3} &= \phi_1(3) X_{j4+2} + a_{j4+3} \\ X_{j4+4} &= \phi_1(4) X_{j4+3} + a_{j4+4} \end{aligned}\right\} \quad (1.3)$$

In practice, there are several motivations for using PARMA models instead of the seasonal ARIMA model for modeling seasonal time series. The first motivation is that periodic models have to do with seasonal adjustment (Franses and Paap, 2004). The second motivation is that the PARMA model is not a homogeneous model for all seasons (as the case for seasonal ARIMA models), it is rather designed for seasonal time series in a more natural way so that each season has its own equation (Smadi, 2002).

Also, there are many differences between seasonal ARIMA and PARMA models. For instance, using an appropriate differencing for the seasonal ARIMA model will make it stationary, while differencing the series of PARMA model produces again another PARMA model.

PARMA models are not stationary in the ordinary weak sense. They are rather examined for a weaker type of stationarity named as periodic stationarity. This means that the mean and the variance of the time series are constants for each season and periodic with period ω and the autocovariance function depends on the time lag and season only. The periodic stationarity of any PARMA model can be examined using the lumped process $Y_j = (X_{j\omega+1}, X_{j\omega+2}, \ldots, X_{j\omega+\omega})^T$, which in fact follows an ω-variate ARMA model (Ula and Smadi, 1997). For instance, the periodic stationarity condition for the $\mathrm{PAR}_{\omega}(1)$ model, exemplified in equation (1.2), reduces to $\left|\prod_{\nu=1}^{\omega} \phi_1(\nu)\right| < 1$ (Obeysekera and Salas,1986).

1.4 Aims of the Study

The aim of this study is to develop a robust estimator for the seasonal autocorrelation function of periodic autoregressive models. This requires mathematical derivations and manipulations. Besides, we will use Monte-Carlo simulation for the comparison and evaluation of various estimators of SACF.

Many factors will be investigated in the course of the study, including the type of PAR model, namely PAR(1), PAR(p) and varying orders PAR models. Also the effect of changing realization length of the time series, N as well as the period length, ω, and the place of the additive outlier on the estimators for the SACF are investigated .

1.5 Overview

The general forms and the stationary conditions the of ARMA model and its extension the PARMA model, are discussed in chapter one. In chapter two, we discussed ACF and partial autocorrelation function (PACF) of ARMA as well as seasonal autocorrelation function (SACF) and seasonal partial autocorrelation function (SPACF) of PARMA models, also we discussed their roles in identification of ARMA models and PARMA models respectively and some general and sampling properties.

In chapter three, we generalize the work of Berkoun et al. (2003) for finding a robust estimators for AR(1) to find a robust estimator for SACF in the case of PAR(1) model. Afterward we find a special formula for finding SACF. We use the Monte-Carlo simulation to identify the most appropriate robust estimator via bias and MSE. The generalization of the work of chapter three to higher orders PAR model is done in chapter four including generalizing the formula of Berkoun et al. (2003). In chapter five we applied our work on some real time series data.

Finally, in chapter six we summarized our results and suggest some problems that deserve further investigation and study.

CHAPTER 2

The Autocorrelation and Seasonal Autocorrelation Functions

2.1 Introduction

In time series analysis, one of the most important steps is to identify a suitable model based on an available realization. For ARMA models, the autocorrelation function (ACF) and partial auto correlation function (PACF) play a primary role in the identification method (Box et al., 1994). Similarly, for PARMA models, two analogous functions; namely the seasonal autocorrelation function (SACF) and the seasonal partial autocorrelation function (SPACF) are utilized for the identification of the seasons orders of PARMA models (Hiple and Mcleod, 1994).

In this chapter, we will review various issues of the ACF and PACF for a stationary time series as well as SACF and SPACF for a periodic stationary time series.

2.2 ACVF and ACF of Stationary Time Series

Downbelow, we assume that $\{X_t : t = 0, \pm1, \pm2, \ldots\}$ is a stationary stochastic process.

Definition. 2.1: The mean function of $\{X_t\}$ is defined as:

$$E(X_t) = \mu$$

for all t.

Definition. 2.2: The autocovariance function (ACVF) of $\{X_t\}$ is (Cryer and Chan, 2008):

$$\gamma_k = Cov(X_t, X_{t-k}) = E[(X_t-\mu)(X_{t-k}-\mu)].$$

Note that for k=0, $\gamma_0 = Cov(X_t, X_t) = Var(X_t)$.

Definition. 2.3: The autocorrelation function (ACF) of $\{X_t\}$ is defined as (Cryer and Chan, 2008):

$$\rho_k = \frac{Cov(X_t, X_{t-k})}{\sqrt{Var(X_t)Var(X_{t-k})}} = \frac{\gamma_k}{\gamma_0} \;; |k| = 0,1,2,...$$

In general the covariance and the correlation are two measures of linear association between two random variables. However, the correlation is usually preferred because it is unitless and easier to interpret. Similarly, for stationary stochastic process, the ACF is usually preferred. The graph of k vs ρ_k; $k = 0,1,2,...$ which is known as the correlogram is a very important graph for model building and identification.

For a stationary process $\{X_t\}$ the ACVF and the ACF have the following properties (Wei,1990):

1. $\gamma_0 = Var(X_t)$; $\rho_0 = 1$.

2. $|\gamma_k| \le \gamma_0$; $|\rho_k| \le 1$.

3. $\gamma_k = \gamma_{-k}$; $\rho_k = \rho_{-k}$, for all k.

4. γ_k and ρ_k are positive semi-definite functions.

For various ARMA models, there are closed formulas for the ACF. Besides, there are general patterns in ACF that are usually used in practice for identification ARMA models. Table (2.1) below summarizes the ACVF and ACF of some simple ARMA models.

Table (2.1): The ACVF and ACF of some simple ARMA models

Model	ACVF	ACF
White Noise	$\gamma_k = \begin{cases} \sigma_a^2 & k=0 \\ 0 & k>0 \end{cases}$.	$\rho_k = \begin{cases} 1 & k=0 \\ 0 & k>0 \end{cases}$.
AR(1)	$\gamma_k = \begin{cases} \dfrac{\sigma_a^2}{1-\phi^2} & k=0 \\ \dfrac{\sigma_a^2}{1-\phi^2}\phi^k & k\geq 1 \end{cases}$.	$\rho_k = \begin{cases} 1 & k=0 \\ \phi^k & k\geq 1 \end{cases}$.
MA(1)	$\gamma_k = \begin{cases} (1+\theta^2)\sigma_a^2 & k=0 \\ -\theta\sigma_a^2 & k=1 \\ 0 & k\geq 2 \end{cases}$.	$\rho_k = \begin{cases} 1 & k=0 \\ \dfrac{-\theta}{1+\theta^2} & k=1 \\ 0 & k\geq 2 \end{cases}$.
ARMA(1,1)	$\gamma_k = \begin{cases} \dfrac{1+\theta^2-2\phi\theta}{1-\phi^2}\sigma_a^2 & k=0 \\ \dfrac{(1-\phi\theta)(\phi-\theta)}{1-\phi^2}\sigma_a^2 & k=1 \\ \phi\gamma_{k-1} & k\geq 2 \end{cases}$.	$\rho_k = \begin{cases} 1 & k=0 \\ \dfrac{(1-\theta\phi)(\phi-\theta)}{1-2\theta\phi+\theta^2}\phi^{k-1} & k\geq 1 \end{cases}$

An important fact regarding the ACF of ARMA models is that for MA(q) model the ACF cuts-off after lag q, while it has no cut-off for AR or mixed ARMA models. Also, the ACF of white noise process cuts-off after lag zero. This fact in practice is usually used to examine whether any time series is a white noise process or not. It is also used for examining independence among residuals of fitted ARMA models.

For a detailed account on ACF of various ARMA models and its properties, see Cryer and Chan (2008).

2.3 PACF of Stationary Time Series

Another important function that is usually estimated for a stationary time series is called the partial autocorrelation function (PACF), which is used to investigate the correlation between X_t and X_{t+k} after removing their mutual linear dependency on their intervening variables X_{t+1} ,X_{t+2},…,X_{t+k-1}. The computation of this function depends mainly on the ACF.

Definition.2.4: let $\{X_t\}$ be a stationary stochastic process, then the partial autocorrelation between X_t and X_{t+k} is:

$$\phi_{kk} = \frac{Cov\left(X_t - \hat{X}_t, X_{t+k} - \hat{X}_{t+k}\right)}{\sqrt{Var(X_t - \hat{X}_t)}}; \; k = 0,1,2,...$$

where:

$$\hat{X}_t = \alpha_1 X_{t+1} + \alpha_2 X_{t+2} + ... + \alpha_{k-1} X_{t+k-1}$$

and,

$$\hat{X}_{t+k} = \alpha_1 X_{t+k-1} + \alpha_2 X_{t+k-2} + ... + \alpha_{k-1} X_{t+1}.$$

where $\hat{X}_t$ and $\hat{X}_{t+k}$ are the best linear predictors of X_t and X_{t+k} respectively in view of X_{t+1},…,X_{t+k-1} and $\alpha_i \left(1 \le i \le k-1\right)$ are the mean squared linear regression coefficients (Wei,1990).

Also, ϕ_{kk} satisfies:

$$\phi_{kk} = \frac{\begin{vmatrix} 1 & \rho_1 & \rho_2 & \cdots & \rho_{k-2} & \rho_1 \\ \rho_1 & 1 & \rho_1 & \cdots & \rho_{k-3} & \rho_2 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ \rho_{k-1} & \rho_{k-2} & \rho_{k-3} & \cdots & \rho_1 & \rho_k \end{vmatrix}}{\begin{vmatrix} 1 & \rho_1 & \rho_2 & \cdots & \rho_{k-2} & \rho_{k-1} \\ \rho_1 & 1 & \rho_1 & \cdots & \rho_{k-3} & \rho_{k-2} \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ \rho_{k-1} & \rho_{k-2} & \rho_{k-3} & \cdots & \rho_1 & 1 \end{vmatrix}}; \quad k = 2,\ldots \qquad (2.1)$$

Another method for the computation of the PACF, ϕ_{kk}, is a recursive method given independently by Durbin (1960) and Levinson (1947) (see also, Cryer and Chan, 2008).

The PACF is also important for identification the suitable ARMA models. It is usually used together with the ACF to identify tentative ARMA models. The PACF of AR(p) models has a cut-off after lag p whereas it has no cut-off for MA or mixed ARMA models. For detailed account on PACF for ARMA models see Wei (1990).

Table (2.2) summarizes general patterns of ACF and PACF for various types of ARMA models. These patterns are usually used in practice to identify suitable ARMA models.

Table (2.2): General behaviors of ACF and PACF for ARMA process

	WN Process	AR(p) Process	MA(q) Process	ARMA(p,q)
ACF	Cuts off after lag zero.	Tails off.	Cuts off after lag q.	Tails off.
PACF	Cuts off after lag zero.	Cuts off after lag p.	Tails off.	Tails off.

2.4 Sample ACF and Sample PACF of Stationary Time Series

Let $\{X_t\}$ be a stationary stochastic process and $\{X_1,...,X_n\}$ be an observed realization of $\{X_t\}$. Then, the mean of the stochastic process $E(X_t)=\mu$ is estimated by the sample mean (Box et al., 1994):

$$\overline{X}=\frac{1}{n}\sum_{t=1}^{n}X_t.$$

Also, the variance of $\{X_t\}$; $\gamma_0 = Var(X_t)$ is estimated by:

$$\hat{\gamma}_0 = C_0 = \frac{1}{n}\sum_{t=1}^{n}(X_t-\overline{X})^2.$$

Similarly; the sample ACVF, $\hat{\gamma}_k$, is given by:

$$\hat{\gamma}_k = C_k = \frac{\sum_{t=1}^{n-k}(X_t-\overline{X})(X_{t+k}-\overline{X})}{n}, \quad k=0,1,2,...$$

and finally the sample ACF is defined as:

$$\hat{\rho}_k = r_k = \frac{C_k}{C_0} = \frac{\sum_{t=1}^{n-k}(X_t-\overline{X})(X_{t+k}-\overline{X})}{\sum_{t=1}^{n}(X_t-\overline{X})^2}, \quad k=0,1,2,.... \qquad (2.2)$$

The above estimators of the mean, variance, γ_k and ρ_k are known as the moment estimators (Wei, 1990).

For stationary Gaussian processes, Bartlett (1946) has shown that for k >0 and k+j>0:

$$Cov(r_k, r_{k+j}) \approx \frac{1}{n}\sum_{i=-\infty}^{\infty}(\rho_i\rho_{i+j} + \rho_{i+k+j}\rho_{i-k}$$
$$-2\rho_k\rho_i\rho_{i-k-j} - 2\rho_{k+j}\rho_i\rho_{i-k} + 2\rho_k\rho_{k+j}\rho_i^2) \qquad (2.3)$$

so that,

$$Var(r_k) \approx \frac{1}{n}\sum_{i=-\infty}^{\infty}\left(\rho_i^2 + \rho_{i+k}\rho_{i-k} - 4\rho_k\rho_i\rho_{i-k} + 2\rho_k^2\rho_i^2\right).$$

(2.4)

For the white noise process, equations (2.3) and (2.4) give:

$$Var(r_k) \approx \frac{1}{n} \quad \text{and} \quad \mathrm{Corr}(r_{k_1}, r_{k_2}) \approx 0 \quad \text{for } k_1 \neq k_2.$$

Besides, for a process following MA(q) model, (2.4) reduces to:

$$Var(r_k) \approx \frac{1}{n}\left(1 + 2\sum_{j=1}^{q}\rho_j^2\right), \qquad \text{for } k > q.$$

The results in (2.3) and (2.4) are obtained asymptotically. For large n, r_k is approximately normally distributed with mean ρ_k and variance given by (2.4). However, it is observed that for white noise and small time lags 1/n over estimates $Var(r_k)$, while r_k and r_j can be highly correlated for small lags k and j.

To obtain the sample PACF, denoted by $\hat{\phi}_{kk}$, we can use Durbin-Levinsion recursive formula with $\hat{\rho}_k$ replacing ρ_k which is given by (Cryer and Chan, 2008):

$$\hat{\phi}_{k,k} = \frac{r_k - \sum_{j=1}^{k-1}\hat{\phi}_{k-1,j}r_{k-j}}{1 - \sum_{j=1}^{k-1}\hat{\phi}_{k-1,j}r_j},$$

with

$$\hat{\phi}_{11} = r_1 \quad \text{and} \quad \hat{\phi}_{k,j} = \hat{\phi}_{k-1,j} - \hat{\phi}_{k,k}\hat{\phi}_{k-1,k-j}, \qquad j = 1,\ldots, k-1.$$

The sample ACF and sample PACF play an important role in the Box-Jenkins methodology for building ARMA models, especially for the identification and estimation phases. Previously, we have summarized general patterns of ACF and PACF of various ARMA models.

The MA(q) model can be identified via the sample ACF, r_k. If r_k falls within the 95% confidence bounds $\pm 2\,\text{S.E}(r_k) = \pm 2\sqrt{\frac{1}{n}\left(1+2\sum_{j=1}^{q} r_j^2\right)}$ for k>q. Notice that, the confidence bounds for identifying MA(q) models get wider as q gets larger. In practice, if all r_k falls within $\pm 2/\sqrt{n}$ then a white noise model is identified (Cryer and Chan, 2008).

Quenoulle (1949) has shown that, under the hypothesis that an AR(p) model is correct, the sample PACF at lag greater than p are approximately normally distributed with mean zero and variance $1/n$. Thus for k>p, to test the null hypothesis that an AR(p) model is correct, we can use $\pm 2/\sqrt{n}$ as 95% confidence bounds (Cryer and Chan, 2008).

Also, the sample ACF may indicate non-stationarity of the time series due to the presence of trend and/or seasonality. In this case, the time series is usually differenced so that it becomes stationary. Therefore, the ACF is sketched again to identify a suitable ARMA model (Wei, 1990).

Another important role of the ACF is that it is used for moment estimation of parameters of ARMA models.

For example, consider the AR(2) model, it can be shown that,

$$\rho_1 = \phi_1 + \rho_1\phi_2 \quad \text{and} \quad \rho_2 = \rho_1\phi_1 + \phi_2 .$$

Thus, the moment estimates of ϕ_1 and ϕ_2 are obtained by replacing ρ_1 and ρ_2 with r_1 and r_2, respectively, then solving for ϕ_1 and ϕ_2. This gives (Cryer and Chan, 2008):

$$\hat{\phi}_1 = \frac{r_1(1-r_2)}{1-r_1^2} \quad \text{and} \quad \hat{\phi}_2 = \frac{r_2 - r_1^2}{1-r_1^2} .$$

2.5 Seasonal ACF and Seasonal PACF

Let $\{X_{j\omega+v}\}$ be a periodic stationary process, then two functions similar to ACF and PACF are developed here and named as the seasonal autocorrelation function (SACF) and seasonal partial autocorrelation function (SPACF). The seasonal ACVF (SACVF) is defined as (Franses and Paap, 2004):

$$\begin{aligned}\gamma_k(v) &= Cov(X_{j\omega+v}, X_{j\omega+v-k}) \\ &= \mathrm{E}\left[\left(X_{j\omega+v} - \mu_v\right)\left(X_{j\omega+v-k} - \mu_{v-k}\right)\right]\end{aligned}$$

where k=0,1,... is the time lag, ω is the period and $v=1,2,\ldots,\omega$ stands for the season. Note that for k=0, $\gamma_0(v)$ denote the variance of the process for season v.

The seasonal ACF (SACF) which depends on the time lag and season only, is defined as (Franses and Paap, 2004):

$$\rho_k(v) = Corr(X_{j\omega+v}, X_{j\omega+v-k}) = \frac{\gamma_k(v)}{\sqrt{\gamma_o(v)\gamma_o(v-k)}}; \; v = 1, 2, \ldots, \omega.$$

Similarly, the seasonal PACF (SPACF) is defined as (Cryer and Chan, 2008):

$$\phi_{kk}(v) = Corr\left(X_{jw+v}, X_{jw+v-k} \middle| X_{jw+v-1}, \ldots, X_{jw+v-k+1}\right)$$

$$\phi_{kk}(\nu) = Corr(\hat{e}_{jw+\nu}, \hat{e}_{jw+\nu-k})$$

where $\hat{e}_{jw+\nu} = X_{jw+\nu} - \hat{X}_{jw+\nu}$, $\hat{e}_{jw+\nu-k} = X_{jw+\nu-k} - \hat{X}_{jw+\nu-k}$ and $\hat{X}_{jw+\nu}$ and $\hat{X}_{jw+\nu-k}$ are the best MSE predictors of $X_{jw+\nu}$ and $X_{jw+\nu-k}$ based on the set $\{X_{jw+\nu-1}, \ldots, X_{jw+\nu-k+1}\}$, respectively.

Now, to define the sample SACF for periodic processes, let $\{X_1, \ldots, X_{n\omega}\}$ be an observed realization of $\{X_{j\omega+\nu}\}$.

Definition. 2.5: The sample seasonal autocovariance function $C_k(\nu)$ is given by (Ula and Smadi, 2003):

$$\hat{\gamma}_k(\nu) = C_k(\nu) = \frac{\sum_{j=0}^{n-1}(X_{j\omega+\nu} - \overline{X}_\nu)(X_{j\omega+\nu-k} - \overline{X}_{\nu-k})}{n}$$

and the sample periodic variance is:

$$\hat{\gamma}_0(\nu) = C_o(\nu) = \frac{\sum_{j=0}^{n-1}(X_{j\omega+\nu} - \overline{X}_\nu)^2}{n},$$

where $\overline{X}_\nu$ is the sample mean of the time series in season ν and n is the number of years of data.

Definition. 2.6: The moment estimate of $\rho_k(\nu)$, or the sample SACF, is given by (Ula and Smadi, 2003):

$$\hat{\rho}_k(\nu) = r_k(\nu) = \frac{\hat{\gamma}_k(\nu)}{\sqrt{\hat{\gamma}_0(\nu)\ \hat{\gamma}_0(\nu-k)}} = \frac{C_k(\nu)}{\sqrt{C_0(\nu)\ C_0(\nu-k)}}. \tag{2.5}$$

If $\{X_{j\omega+\nu}\}$ is Gaussian, i.e., if the white noise terms are independent and normally distributed, Pagano (1978) showed that, for all ν and k, $\hat{\gamma}_k(\nu)$ is consistent estimator for $\hat{\gamma}_k(\nu)$. It is also known that $\overline{X}_\nu$ is an unbiased and consistent estimator of μ_ν under periodic stationarity conditions.

For large n, replacing the sample means and variances, $\overline{X}_\nu$ and $\hat{\gamma}_0(\nu)$ in $r_k(\nu)$ with μ_ν and $\gamma_0(\nu)$ respectively, then,

$$\mathrm{E}(r_k(\nu)) \approx \frac{n-\varphi(\nu,k,\omega)}{n}\rho_k(\nu),$$

where $\varphi(\nu,k,\omega) = [[k-\nu/\omega]]+1$ and $[[x]]$ stands for the greatest integer below or equal to x. Since $\varphi(\nu,k,\omega)$ is a fixed quantity for ν and ω, then $r_k(\nu)$ is asymptotically unbiased estimator for $\rho_k(\nu)$ (Ula and Smadi, 2003).

Also, under the condition $E(X_{j\omega+\nu}) = 0$ and for $k_2 \geq k_1 \geq 0$, we have,

$$Cov(r_{k_1}(\nu), r_{k_2}(\nu)) \approx \frac{1}{n}\sum_{m=a-(n-1)}^{n-1-b}\left[1-\frac{\eta(m)+b}{n}\right]\{\rho_{m\omega}(\nu)\rho_{m\omega+k_2-k_1}(\nu-k_1) + \rho_{m\omega+k_2}(\nu)\rho_{m\omega-k_1}(\nu-k_1)\} \qquad (2.6)$$

where,

$$\eta(m) = \begin{cases} a-b-m, & a-(n-1) \leq m < a-b \\ 0, & a-b \leq m \leq 0 \\ m, & m > 0 \end{cases},$$

$a=\varphi(\nu,k_1,\omega)$ and $b=\varphi(\nu,k_2,\omega)$. if $k_1=k_2=k$ and for large n, (2.6) becomes (Ula and Smadi, 2003):

$$Var(r_k(\nu))\approx\frac{1}{n}\left[1+[\rho_k(\nu)]^2+2\sum_{m=1}^{\infty}\{\rho_{mw}(\nu)\rho_{mw}(\nu-k)+\rho_{mw+k}(\nu)\rho_{mw-k}(\nu-k)\}\right] \quad . \quad (2.7)$$

Thus, for the seasonal white noise process, (2.7) reduces to:

$$Var(r_k(\nu))\approx\frac{1}{n}\ .$$

Sakai (1982) proposed an algorithm for computing the SPACF, $\phi_{kk}(\nu)$, iteratively. Besides, Sakai (1982) proved (under the assumption of white noise terms being independent and normal) that similar to ordinary PACF for AR models, if a season ν follows an AR(p(ν)) model, then for all ν and $k>p(\nu)$, the SPACF are asymptotically independent for each k and ν, and normally distributed with mean zero and variance 1/n. This result is helpful for the identification of PAR models.

Finally, the identification of PARMA models follows a similar approach to that of ARMA models. The main difference is that for a periodic time series with period ω, we sketch ω pairs of graphs for the sample SACF and sample SPACF for each season separately. Then, for season ν, a cut-off in SACF usually indicates a MA model and a cut-off in SPACF indicates an AR model. Moreover, if all values of SACF and SPACF falls within $\pm 2/\sqrt{n}$, then a white noise model is identified.

2.6 The Sample SACF of Seasonal White Noise Process

We have seen that for seasonal white noise process the variance of the sample SACF for each season asymptotically equals 1/n. In this section, if the process is a white noise process, we will use Monte-Carlo simulation to investigate the accuracy of $1/n$ as an estimate of the Variance of $r_k(\nu)$ for different realizations length.

Therefore we are going to generate data $\{X_1,...,X_{n\omega}\}$ from a seasonal white noise process for period equal four. Then using the simulated data to calculate the sample SACF and their variances.

As far as the simulation-work is concerned; 1000 realizations each of length n years (n= 30, 50, 100) will be simulated from white noise process that is assumed normal with mean zero and periodic variances σ_ν^2. In each simulated realization, the estimate of $\rho_k(\nu)$ is computed, namely $r_k(\nu)$ as given by (2.5). Then, based on the 1000 iterations, the mean and variance of $r_k(\nu)$ are computed. Also, the ratio of the variance of $r_k(\nu)$ to 1/n is obtained. Here, we will consider the seasonal white noise model with $\omega = 4$; $\sigma_\nu^2 = 1, 64, 4, 9$, denoted as Model 1.

Tables (2.3)-(2.5) and Figure (2.1) present the results. Figure (2.1) presents the mean of $r_k(\nu)$ for Model 1 when n=30, 50, 100. Tables (2.3), (2.4) and (2.5) summarize the variances and differences between the variances of $r_k(\nu)$ and 1/n (given in brackets) of $r_k(\nu)$ for white noise process of Model 1 where n=30, 50, 100.

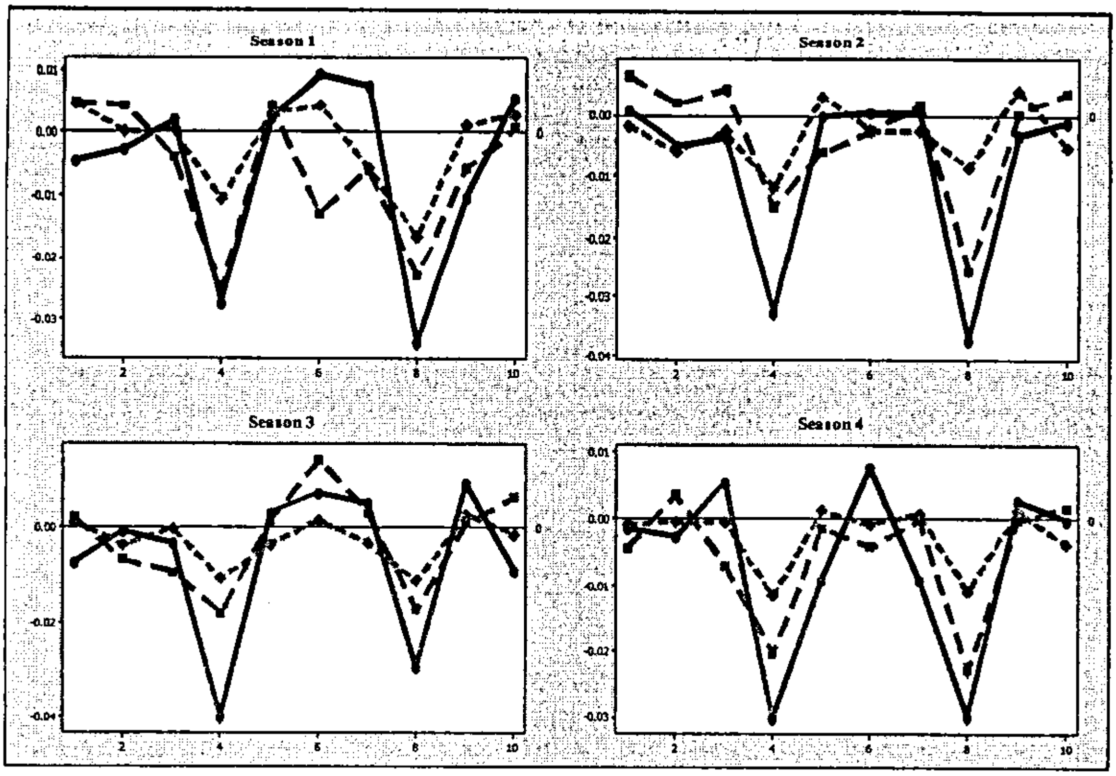

Figure (2.1): The mean for $r_k(\nu)$ for Model 1 (—: n=30, – – : n=50, ---: n=100)

Table (2.3): The variance and differences with respect to 1/n (given in brackets) for $r_k(\nu)$ of white noise process of Model 1 and n=30

	Season			
Lag	**1**	**2**	**3**	**4**
1	0.0332 (-0.0002)	0.0357 (0.0024)	0.0344 (0.0010)	0.0347 (0.0014)
2	0.0324 (-0.0009)	0.0347 (0.0014)	0.0349 (0.0015)	0.0340 (0.0007)
3	0.0343 (0.0010)	0.0333 (-0.0001)	0.0313 (-0.0021)	0.0346 (0.0012)
4	0.0279 (-0.0055)	0.0286 (-0.0047)	0.0298 (-0.0035)	0.0298 (-0.0036)
5	0.0288 (-0.0045)	0.0306 (-0.0027)	0.0360 (0.0027)	0.0345 (0.0012)
6	0.0315 (-0.0018)	0.0332 (-0.0001)	0.0303 (-0.0030)	0.0332 (-0.0001)
7	0.0346 (0.0012)	0.0311 (-0.0023)	0.0314 (-0.0019)	0.0338 (0.0005)
8	0.0303 (-0.0031)	0.0291 (-0.0043)	0.0295 (-0.0038)	0.0299 (-0.0035)
9	0.0317 (-0.0017)	0.0337 (0.0004)	0.0328 (-0.0005)	0.0325 (-0.0008)
10	0.0304 (-0.0029)	0.0305 (-0.0028)	0.0342 (0.0009)	0.0298 (-0.0035)

Table (2.4): The variance and differences with respect to 1/n (given in brackets) for $r_k(\nu)$ of white noise process of Model 1 and n=50

	Season			
Lag	**1**	**2**	**3**	**4**
1	0.0203 (0.0003)	0.0199 (-0.0001)	0.0205 (0.0005)	0.0203 (0.0003)
2	0.0194 (-0.0006)	0.0210 (0.0010)	0.0220 (0.0020)	0.0202 (0.0002)
3	0.0195 (-0.0005)	0.0203 (0.0003)	0.0200 (0.0000)	0.0228 (0.0028)
4	0.0179 (-0.0021)	0.0173 (-0.0027)	0.0200 (0.0000)	0.0182 (-0.0018)
5	0.0198 (-0.0002)	0.0217 (0.0017)	0.0194 (-0.0006)	0.0212 (0.0012)
6	0.0199 (-0.0001)	0.0197 (-0.0003)	0.0194 (-0.0006)	0.0197 (-0.0003)
7	0.0194 (-0.0006)	0.0184 (-0.0016)	0.0205 (0.0005)	0.0200 (0.0000)
8	0.0188 (-0.0012)	0.0182 (-0.0018)	0.0186 (-0.0014)	0.0193 (-0.0007)
9	0.0190 (-0.0010)	0.0188 (-0.0012)	0.0188 (-0.0012)	0.0179 (-0.0021)
10	0.0193 (-0.0007)	0.0193 (-0.0007)	0.0197 (-0.0003)	0.0174 (-0.0026)

Table (2.5): The variance and differences with respect to 1/n (given in brackets) for $r_k(\nu)$ of white noise process of Model 1 and n=100

	Season			
Lag	**1**	**2**	**3**	**4**
1	0.0105 (0.0005)	0.0100 (0.0000)	0.0101 (0.0001)	0.0103 (0.0003)
2	0.0092 (-0.0008)	0.0108 (0.0008)	0.0098 (-0.0002)	0.0101 (0.0001)
3	0.0098 (-0.0002)	0.0102 (0.0002)	0.0101 (0.0001)	0.0112 (0.0012)
4	0.0097 (-0.0003)	0.0099 (-0.0001)	0.0093 (-0.0007)	0.0098 (-0.0002)
5	0.0095 (-0.0005)	0.0097 (-0.0003)	0.0092 (-0.0008)	0.0097 (-0.0003)
6	0.0096 (-0.0004)	0.0105 (0.0005)	0.0090 (-0.0010)	0.0094 (-0.0006)
7	0.0101 (0.0001)	0.0103 (0.0003)	0.0105 (0.0005)	0.0099 (-0.0001)
8	0.0091 (-0.0009)	0.0101 (0.0001)	0.0096 (-0.0004)	0.0097 (-0.0003)
9	0.0102 (0.0002)	0.0096 (-0.0004)	0.0100 (0.0000)	0.0097 (-0.0003)
10	0.0100 (0.0000)	0.0092 (-0.0008)	0.0093 (-0.0007)	0.0103 (0.0003)

From Figure (2.1) we can see that the mean of $r_k(\nu)$ of the white noise process closes to zero as n increases for all seasons.

From Tables (2.3)-(2.5), we can see that the variances of $r_k(\nu)$ of the white noise process are fairly close to 1/n. For instance, in Table (2.3) the variances are close to 1/n= 1/30= .033. Also, in Tables (2.3), (2.4) and (2.5) the differences between Var($r_k(\nu)$) and 1/n are very close to zero for all seasons specially as n increases.

CHAPTER 3

Robust Estimation of SACF of PAR (1) Model

3.1 Introduction

Estimation and test procedures are said to be robust if they are little influenced by blatant departures from assumption. Those procedures aim to minimize the influence of outliers or anomalous observations when these lead to break down in basic assumptions while performing at the same time as well as the optimum methods when assumptions hold (Sprent and Smeeton, 2001).

It is known that time series data may be contaminated with outliers which affect all the stages of time series analysis, such as the model identification, estimation and forecasting.

For time series, two kinds of outliers can be distinguished, namely additive outliers and innovative outliers. These two kinds of outliers are often abbreviated as AO and IO, respectively.

An additive outlier occurs at time T if the underlying process is perturbed additively at time T, so that the data equal

$$Y_t' = Y_t + \Delta_A P_t(T)$$

where $\{Y_t\}$ is the uncontaminated process, $\{Y_t'\}$ denotes the observed, contaminated process that is affected by the outlier, Δ_A is the magnitude of the isolated, additive outlier and $P_t(T)$ is the pulse function,

$$P_t(T) = \begin{cases} 1, & \text{t = T} \\ 0, & \text{Otherwise} \end{cases}$$

On the other hand, an innovative outlier occurs at time T is an event whose effect is propagated according to the structure of the model of Y_t. Thus, the innovative model is,

$$Y_t' = Y_t + \Delta_I I_t(T)$$

where $\{Y_t\}$ is the uncontaminated process, $\{Y_t'\}$ denotes the observed, contaminated process that is affected by the outlier, Δ_I is the magnitude of the innovative outlier and

$$I_t(T) = \begin{cases} 1, & t \geq T \\ 0, & t < T \end{cases}$$

Thus, an innovative outliers at T perturb all observations on and after T.

The impact of outliers on the parameter estimation of ARIMA models has been studied by several researchers including but not limited to Denby and Martin (1979), Chang and Tiao (1983), Martin and Yohai (1986), Pena (1983, 1990, 1991), Barnett and Lewis (1994), Bianco et al. (1996) and Mira and Sanchez (2003).

Several robust estimation procedures for ARMA model parameters have been proposed along the line of Huber (1964) for location parameters. Denby and Martin (1979) proposed the genelized M-estimates for autoregressive processes and Bustos and Yohai (1986) took the autocovariance structure of time series into consideration when robustifying the estimators. While robust estimation for PAR models is rarely studied, Shao (2007) proposed a robust estimation method for PAR(p) model parameters.

Berkoun et al. (2003) investigated robust inference for serial correlation in AR (1) process in the presence of a single additive outlier. Assuming that $\{Z_1,\ldots,Z_n\}$ is a time series following the zero-mean AR(1) model contaminated with a single additive outlier, they have investigated three estimators for ρ_1, namely:

$$\hat{\rho}_1 = \frac{\sum_{t=2}^{n} Z_t Z_{t-1}}{\sum_{t=2}^{n} Z_{t-1}^2} \tag{3.1}$$

$$\tilde{\rho}_1 = Med\left\{\frac{Z_2}{Z_1}, \frac{Z_3}{Z_2}, \ldots, \frac{Z_n}{Z_{n-1}}\right\} \tag{3.2}$$

$$\breve{\rho}_1 = \frac{Med\{Z_1Z_2, Z_2Z_3, \ldots, Z_{n-1}Z_n\}}{Med\{Z_1^2, Z_2^2, \ldots, Z_{n-1}^2\}} \tag{3.3}$$

where Med{.} stands for the median, $\hat{\rho}_1$ is the ordinary moment estimator of ρ_1 while $\tilde{\rho}_1$ and $\breve{\rho}_1$ are two robust estimators of ρ_1 originally proposed by Hurwicz (1950) and Haddad (2000), respectively.

In this chapter, we are interested in robust estimation for the SACF under PAR models. More specifically, we interest to generalize the work of Berkoun et al. (2003) for $PAR_\omega(1)$ and other PAR models, then we are going to estimate the SACF, $\rho_k(\nu)$ for various seasons and time lags.

3.2 Robust Estimation of $\rho_1(\nu)$ in PAR(1) Model

Assume that $\{X_{t,\nu}\}$ follows $PAR_\omega(1)$ model as defined in (1.3) and define $\{Z_{t,\nu}\}$ to be the same as $\{X_{t,\nu}\}$ but contaminated with an additive outlier Δ at year t_0 and season ν_0, i.e,

$$Z_{t,\nu} = \begin{cases} X_{t,\nu}, & (t,\nu) \neq (t_0,\nu_0) \\ X_{t,\nu} + \Delta, & (t,\nu) = (t_0,\nu_0) \end{cases}$$

Accordingly, we want to estimate the SACF of $\{Z_{t,\nu}\}$; i.e. $\rho_k(\nu)$; $k \geq 1$.

Smadi et al. (2009) investigated robust estimates of the first lag SACF of $\text{PAR}_\omega(1)$ which denoted by $\rho_1(\nu)$. Corresponding to the estimates of ρ_1 under AR(1) model which are considered by Berkoun et al. (2003) and given by (3.1)-(3.3), Smadi et al. (2009) proposed the following estimates of $\rho_1(\nu)$:

$$\hat{\rho}_1(\nu) = \frac{\sum_{t=1}^{n}(Z_{t,\nu} - \bar{Z}_\nu)(Z_{t,\nu-1} - \bar{Z}_{\nu-1})}{\sqrt{\sum_{t=1}^{n}(Z_{t,\nu} - \bar{Z}_\nu)^2 \sum_{t=1}^{n}(Z_{t,\nu-1} - \bar{Z}_{\nu-1})^2}} \tag{3.4}$$

$$\tilde{\rho}_1(\nu) = \underset{t}{Med}\left\{\frac{Z^*_{t,\nu}}{Z^*_{t,\nu-1}}\right\}\sqrt{\frac{\underset{t}{Med}(Z^{*2}_{t,\nu-1})}{\underset{t}{Med}(Z^{*2}_{t,\nu})}} \tag{3.5}$$

$$\breve{\rho}_1(\nu) = \frac{\underset{t}{Med}\{Z^*_{t,\nu} Z^*_{t,\nu-1}\}}{\sqrt{\underset{t}{Med}\{Z^{*2}_{t,\nu}\}\underset{t}{Med}\{Z^{*2}_{t,\nu-1}\}}} \tag{3.6}$$

where $Z^*_{t,\nu} = Z_{t,\nu} - Med\{Z_{t,\nu}\}, t = 1,\ldots,n$ is the seasonally median subtracted time series and $\underset{t}{Med}\{Z_{t,\nu}\}$ is the median of the data in season $\nu; \nu = 1,\ldots.\omega$.

The estimator $\hat{\rho}_1(\nu)$ in (3.4) is the moment estimator of $\rho_1(\nu)$ which corresponds to the estimator $\hat{\rho}_k(\nu)$ that is given by (2.5) for k=1. Also, the estimator $\tilde{\rho}_1(\nu)$ is a generalization of $\tilde{\rho}_1$ for AR(1) model which is given in (3.2) which makes use of the fact that for $\text{PAR}_\omega(1)$ model

$$\rho_1(\nu) = \phi_1(\nu)\sqrt{\frac{\gamma_0(\nu-1)}{\gamma_0(\nu)}}. \tag{3.7}$$

The estimator $\breve{\rho}_1(\nu)$ in (3.6) is a generalization of $\breve{\rho}_1$ for AR(1) model that is given by (3.3).

By using Monte-Carlo simulation, Smadi et al. (2009) showed that the ordinary moment estimator $\hat{\rho}_1(\nu)$ is affected by the additive outlier while the other two estimators were apparently robust to the existing of outliers. As reported by Smadi et al. (2009), the estimator $\tilde{\rho}_1(\nu)$ was the best estimator among the other considered estimators for almost all considered cases. Therefore, our objective in this chapter is to estimate the SACF, $\rho_k(\nu)$ for all time lags k for the PAR$_\omega$(1) model.

Theorem (3.1): If $\{X_{j\omega+\nu}\}$ follows a periodic stationary PAR$_\omega$(1) model, then the SACF at lag k is:

$$\begin{aligned}\rho_k(\nu) &= \phi_1(\nu)\sqrt{\frac{\gamma_0(\nu-1)}{\gamma_0(\nu)}}\rho_{k-1}(\nu-1)\\ &= \rho_1(\nu)\rho_1(\nu-1)\rho_1(\nu-2)\ldots\rho_1(\nu-k+1).\end{aligned}$$

Proof. It is easy to show that the first lag seasonal autocovariance function, SACVF,

$$\gamma_1(\nu) = Cov(X_{t,\nu}, X_{t,\nu-1}) = Cov(\phi_1(\nu)X_{t,\nu-1} + a_{t,\nu} \quad, X_{t,\nu-1}) = \phi_1(\nu)\gamma_0(\nu-1)$$

then the first lag SACF,

$$\rho_1(\nu) = \frac{\gamma_1(\nu)}{\sqrt{\gamma_0(\nu)\gamma_0(\nu-1)}} = \frac{\phi_1(\nu)\gamma_0(\nu-1)}{\sqrt{\gamma_0(\nu)\gamma_0(\nu-1)}} = \phi_1(\nu)\sqrt{\frac{\gamma_0(\nu-1)}{\gamma_0(\nu)}} \qquad (3.8)$$

and the second lag SACVF,

$$\begin{aligned}\gamma_2(\nu) &= Cov(X_{t,\nu}, X_{t,\nu-2})\\ &= Cov(\phi_1(\nu)X_{t,\nu-1} + a_{t,\nu}, X_{t,\nu-2})\\ &= \phi_1(\nu)\gamma_1(\nu-1)\end{aligned}$$

So that the second lag SACF,

$$\rho_2(\nu)=\frac{\gamma_2(\nu)}{\sqrt{\gamma_0(\nu)\gamma_0(\nu-2)}}$$
$$=\frac{\phi_1(\nu)\gamma_1(\nu-1)}{\sqrt{\gamma_0(\nu)\gamma_0(\nu-2)}}$$
$$=\phi_1(\nu)\frac{\gamma_1(\nu-1)}{\sqrt{\gamma_0(\nu-1)\gamma_0(\nu-2)}}\sqrt{\frac{\gamma_0(\nu-1)\gamma_0(\nu-2)}{\gamma_0(\nu)\gamma_0(\nu-2)}}$$
$$=\phi_1(\nu)\rho_1(\nu-1)\sqrt{\frac{\gamma_0(\nu-1)}{\gamma_0(\nu)}}$$
$$=\phi_1(\nu)\sqrt{\frac{\gamma_0(\nu-1)}{\gamma_0(\nu)}}\rho_1(\nu-1)=\rho_1(\nu)\rho_1(\nu-1) \tag{3.9}$$

and the third lag SACVF,

$$\gamma_3(\nu)=Cov(X_{t,\nu},X_{t,\nu-3})$$
$$=Cov(\phi_1(\nu)X_{t,\nu-1}+a_{t,\nu},X_{t,\nu-3})$$
$$=\phi_1(\nu)\gamma_2(\nu-1)$$

So that the third lag SACF,

$$\rho_3(\nu)=\frac{\gamma_3(\nu)}{\sqrt{\gamma_0(\nu)\gamma_0(\nu-3)}}$$
$$=\frac{\phi_1(\nu)\gamma_2(\nu-1)}{\sqrt{\gamma_0(\nu)\gamma_0(\nu-3)}}$$
$$=\phi_1(\nu)\sqrt{\frac{\gamma_0(\nu-1)}{\gamma_0(\nu)}}\rho_2(\nu-1)$$
$$=\rho_1(\nu)\rho_2(\nu-1)$$
$$=\rho_1(\nu)\rho_1(\nu-1)\rho_1(\nu-2). \tag{3.10}$$

In the same manner of deriving (3.8)-(3.10), $\rho_k(\nu)$ for PAR$_\omega$(1) model can be written as:

$$\rho_k(\nu)=\phi_1(\nu)\sqrt{\frac{\gamma_0(\nu-1)}{\gamma_0(\nu)}}\rho_{k-1}(\nu-1)$$
$$=\rho_1(\nu)\rho_{k-1}(\nu-1) \qquad ;k\geq 1$$

In general,

$$\rho_k(\nu)=\rho_1(\nu)\rho_1(\nu-1)\rho_1(\nu-2)\dots\rho_1(\nu-k+1).$$

Note that in AR(1) model, $\rho_k=\phi^k$ and $\phi=\rho_1$ so $\rho_k=\rho_1^k$. In PAR$_\omega$(1), if the model is non-periodic then $\rho_1(\nu)=\rho_1(\nu-1)=\dots=\rho_1(\nu-k+1)=\rho_1$. Consequently, $\rho_k(\nu)=\rho_1(\nu)\rho_1(\nu-1)\rho_1(\nu-2)\dots\rho_1(\nu-k+1)=\rho_1\dots\rho_1=\rho_1^k$.

To estimate $\rho_k(\nu)$ under the PAR$_\omega$(1) model, we proposed and studied the following estimators:

$$\hat{\rho}_k(\nu)=\hat{\rho}_1(\nu)\hat{\rho}_{k-1}(\nu-1) \qquad (3.11)$$

$$\tilde{\rho}_k(\nu)=\tilde{\rho}_1(\nu)\tilde{\rho}_{k-1}(\nu-1) \qquad (3.12)$$

$$\breve{\rho}_k(\nu)=\breve{\rho}_1(\nu)\breve{\rho}_{k-1}(\nu-1) \qquad (3.13)$$

Where the estimators $\hat{\rho}_1(\nu)$, $\tilde{\rho}_1(\nu)$ and $\breve{\rho}_1(\nu)$ are given by (3.4), (3.5) and (3.6) respectively. So, we can compute (3.11), (3.12) and (3.13) for different time lags.

3.3 Methodology and Simulation Results

In this section we used the Monte-Carlo simulation to generate data from PAR$_\omega$(1) model of several period values and contaminated with some additive outliers. The simulated data are used to calculate the estimators (3.11), (3.12) and (3.13) for different time lags. After that we study the robustness of those estimators on the basis of bias and MSE.

As far as the simulation-work is concerned; 1000 realizations each of length n years (30, 50, 100) will be simulated from $\text{PAR}_{\omega}(1)$ models assuming that the white noise process is normal with mean zero and periodic variances σ_{ν}^{2}. In each realization, some outliers with fixed magnitudes are added at specific times. Now, in each simulated realization the estimate of $\rho_k(\nu)$ is computed, say $r_k(\nu)$. Then, based on the 1000 realizations, the bias mean (or absolute bias mean) and MSE are computed.

In order to investigate the behavior of the three estimators (3.4), (3.5) and (3.6) we will carry out simulation based on the following $\text{PAR}_{\omega}(1)$ models:

(1) $\text{PAR}_4(1)$ with ϕ's : 1.1, -0.8, 0.95, 0.7 and σ_{ν}^{2}'s : 1, 64, 4, 9.

(2) $\text{PAR}_{12}(1)$ with ϕ's : 1.1, -0.8, 0.95, 0.7, 0.4, 1.1, 0.45, 0.33, 0.9, 0.7, 1.2, .77

and σ_{ν}^{2}'s : 1, 64, 4, 9, 1, 16, 4, 4, 36, 16, 9, 36.

The above two PAR models are chosen to be periodic stationary satisfy condition $\left|\prod_{\nu=1}^{\omega}\phi_1(\nu)\right|<1$. Regarding to the existence of the additive outliers we will take the following cases,

Case 1: A single additive outlier (Δ=100) added at t=13 (season one).

Case 2: Two additive outliers (Δ_1=100, Δ_2=80) added at t=13, 14 (seasons one and two respectively).

Case 3: Four additive outliers (Δ_1=100, Δ_2=80, Δ_3=120, Δ_4=90) added at t=13, 14, 15 and 16 (seasons one, two, three and four respectively).

Then we compute the mean absolute bias and MSE as follows:

$$\text{Absolute Bias} = \left| \frac{1}{1000} \sum_{j=1}^{1000} \left((r_k(\nu))_j - \rho_k(\nu) \right) \right|$$

and

$$MSE = \frac{1}{1000} \sum_{j=1}^{1000} \left((r_k(\nu))_j - \rho_k(\nu) \right)^2 .$$

To perform the simulation, the R-package is used with codes written by the author and the supervisor.

In what follows, the models $\text{PAR}_4(1)$ and $\text{PAR}_{12}(1)$ will be referred to as Model 1 and Model 2 respectively.

Due to the large amount of results, we have summarized the most important results in Tables (3.2)-(3.8) and Figures (3.2)-(3.18).

Table (3.1) and Figure (3.1) show the theoretical SACF ($\rho_k(\nu)$) for Model 1. Note that in Figure (3.1) for the four seasons there is a jagged periodic pattern repeated every four lags and the theoretical SACF decays to zero as time lag increases. Recall that in the ordinary AR(1) model the jagged pattern in the theoretical SACF appears if the ϕ value is less than zero. Notice here that for Model 1, $\prod_{\nu=1}^{4} \phi_1(\nu) = -0.5852$ which is negative.

The results corresponding to Model 1 are summarized in Tables (3.3)-(3.5) and Figures (3.3)-(3.12), whereas those of Model 2 are presented in Tables (3.6)-(3.8) and Figures (3.13)-(3.14).

In Table (3.2) we explore the bias mean of the three estimators (3.11), (3.12) and (3.13) of $\rho_k(\nu)$ for Model 1. In view of the results in Table (3.2) it is difficult to compare the three estimators via the bias mean. Besides, our focus here is on the robustness of these estimators not on the nature of their biases. Therefore, for the remaining results we will restrict ourselves to the absolute bias mean.

Figures (3.3) and (3.4) present the absolute bias and MSE of the three estimators of the SACF of Model 1 with no additive outlier and n=100, while Figures (3.5)-(3.10) present the absolute bias and MSE for each estimator separately.

Tables (3.3)-(3.5) presents the absolute bias and MSE (in brackets) of the three estimators of the SACF of Model 1 with additive outlier at season one for n=30, 50 and 100 and Tables (3.6)-(3.8) contain the bias and MSE for the three estimators (3.4), (3.5) and (3.6) for Model 2.

Figures (3.11)-(3.12) present the absolute bias and MSE of the three estimators of the SACF of Model 1 with additive outlier at season one for n=30, while Figures (3.13) and (3.14) present the absolute bias and MSE of the three estimators of the SACF of Model 2 with additive outlier at season one for n=30.

Figures (3.15)-(3.16) present the absolute bias and MSE of the three estimators of the SACF of Model 1 with two additive outlier at seasons one and two for n=30 and Figures (3.17)-(3.18) present the absolute bias and MSE of the three estimators of the SACF of Model 1 with four additive outlier at seasons one, two, three and four for n=30.

In Tables (3.2)-(3.8), the estimators $\hat{\rho}_k(\nu)$, $\tilde{\rho}_k(\nu)$ and $\breve{\rho}_k(\nu)$ which are given by (3.11), (3.12) and (3.13) are abbreviated as $\hat{\rho}_k$, $\tilde{\rho}_k$ and $\breve{\rho}_k$ for simplicity.

Table (3.1): The theoretical values of SACF, $\rho_k(v)$ for $PAR_4(1)$ model

	Season			
Time lag	**1**	**2**	**3**	**4**
0	1.000	1.000	1.000	1.000
1	0.9932	-0.6516	0.9807	0.9221
2	0.9159	-0.6471	-0.6390	0.9043
3	0.8982	-0.5967	-0.6346	-0.5892
4	-0.5852	-0.5852	-0.5852	-0.5852
5	-0.5812	0.3813	-0.5739	-0.5396
6	-0.5360	0.3787	0.3739	-0.5292
7	-0.5256	0.3492	0.3714	0.3448
8	0.3425	0.3425	0.3425	0.3425
9	0.3401	-0.2231	0.3358	0.3158
10	0.3136	-0.2216	-0.2188	0.3097

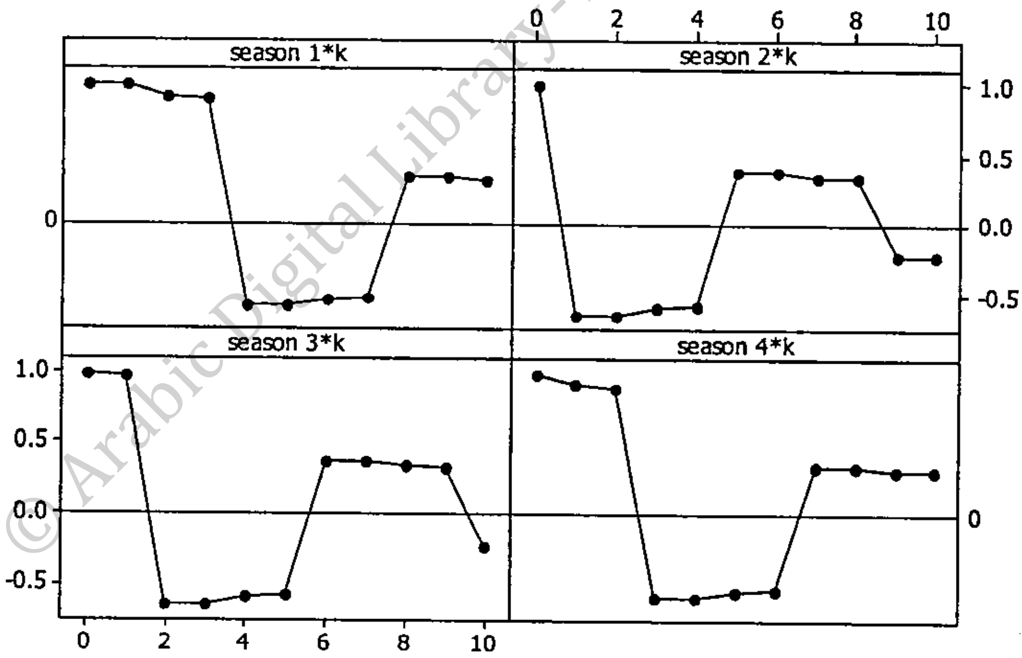

Figure (3.1): The theoretical values of SACF for Model 1

Table (3.2): The bias mean of the three estimators of the SACF of the $PAR_4(1)$ model (Model (1)), n = 100 with additive outlier at season one

	Season											
	1			2			3			4		
Lag	$\hat{\rho}_k$	$\tilde{\rho}_k$	$\breve{\rho}_k$	$\hat{\rho}_k$	$\tilde{\rho}_k$	$\breve{\rho}_k$	$\hat{\rho}_k$	$\tilde{\rho}_k$	$\breve{\rho}_k$	$\hat{\rho}_k$	$\tilde{\rho}_k$	$\breve{\rho}_k$
1	-0.5960	-0.0580	-0.1280	0.3850	0.0455	0.1496	-0.001	-0.0010	-0.017	-0.0060	-0.0030	-0.0530
2	-0.5510	-0.0600	-0.1610	0.0406	0.0837	0.2062	0.3781	0.0478	0.1536	-0.0060	-0.0090	-0.0650
3	-0.5400	-0.0640	-0.1680	0.0419	0.0820	0.2059	0.0409	0.0852	0.208	0.3504	0.0492	0.1592
4	0.4623	0.0857	0.2063	0.0419	0.0857	0.2063	0.0424	0.0857	0.2063	0.0406	0.0857	0.2063
5	0.3603	0.1117	0.2395	-0.2420	-0.0320	-0.1370	0.0432	0.0822	0.2026	0.0448	0.0711	0.1913
6	0.3346	0.0975	0.2214	-0.0650	-0.0530	-0.1570	-0.2390	-0.0320	-0.1340	0.0456	0.0703	0.1874
7	0.3282	0.0963	0.2169	-0.0680	-0.0470	-0.1430	-0.0660	-0.0520	-0.1530	-0.2220	-0.0270	-0.1230
8	-0.3240	-0.0470	-0.1410	-0.0680	-0.0480	-0.1410	-0.0680	-0.0470	-0.1410	-0.0680	-0.0480	-0.1410
9	-0.2260	-0.0620	-0.1530	0.1449	0.0002	0.0731	-0.067	-0.0440	-0.1360	-0.0630	-0.0350	-0.1230
10	-0.2080	-0.0500	-0.1350	0.0461	0.0126	0.0821	0.1417	-0.0000	0.0701	-0.0620	-0.0340	-0.1190

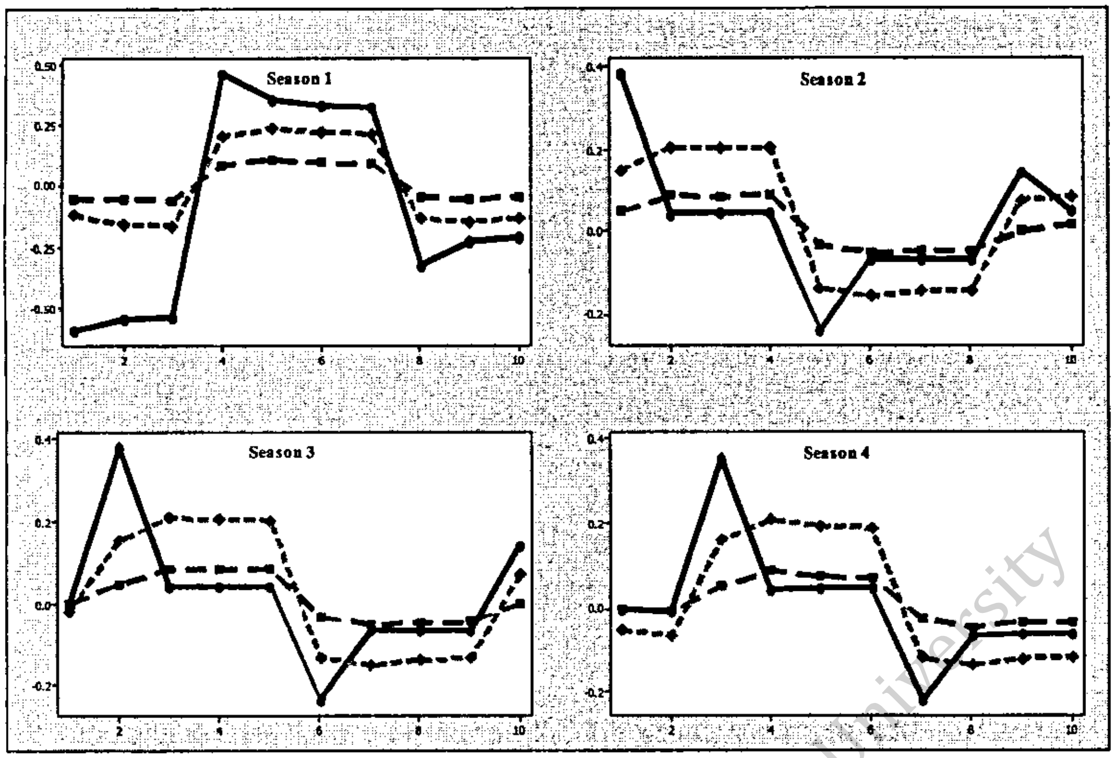

Figure (3.2): The mean bias for the three estimators for $PAR_4(1)$ model (Model 1) with n=30 and a single additive outlier at season one (——: $\hat{\rho}_k(\nu)$, — — : $\tilde{\rho}_k(\nu)$, -----: $\breve{\rho}_k(\nu)$)

Table (3.3): The absolute bias and MSE (in brackets) of the three estimators of the SACF of the $PAR_4(1)$ model (Model 1), n = 100 with no outlier

	Season											
	1			2			3			4		
Lag	$\hat{\rho}_k$	$\tilde{\rho}_k$	$\breve{\rho}_k$	$\hat{\rho}_k$	$\tilde{\rho}_k$	$\breve{\rho}_k$	$\hat{\rho}_k$	$\tilde{\rho}_k$	$\breve{\rho}_k$	$\hat{\rho}_k$	$\tilde{\rho}_k$	$\breve{\rho}_k$
1	0.0104 (0.0002)	0.0020 (0.0033)	0.0290 (0.0037)	0.0063 (0.0050)	0.0065 (0.0170)	0.1351 (0.0399)	0.0004 (0.0000)	0.0013 (0.0045)	0.0121 (0.0049)	0.0019 (0.0004)	0.0061 (0.0078)	0.0477 (0.0129)
2	0.0112 (0.0007)	0.0092 (0.0079)	0.0724 (0.0179)	0.0131 (0.0053)	0.0087 (0.0167)	0.1488 (0.0433)	0.0061 (0.0053)	0.0065 (0.0169)	0.1374 (0.0417)	0.0020 (0.0006)	0.0063 (0.0086)	0.0564 (0.0184)
3	0.0110 (0.0008)	0.0092 (0.0089)	0.0801 (0.0234)	0.0134 (0.0064)	0.0134 (0.0152)	0.1582 (0.0461)	0.0130 (0.0056)	0.0087 (0.0166)	0.1506 (0.0449)	0.0060 (0.0061)	0.0114 (0.0153)	0.1475 (0.0446)
4	0.0121 (0.0063)	0.0141 (0.0141)	0.1588 (0.0473)	0.0135 (0.0066)	0.0141 (0.0141)	0.1588 (0.0473)	0.0129 (0.0067)	0.0141 (0.0141)	0.1588 (0.0473)	0.0126 (0.0065)	0.0141 (0.0141)	0.1588 (0.0473)
5	0.0186 (0.0068)	0.0150 (0.0150)	0.1685 (0.0507)	0.0181 (0.0116)	0.0004 (0.0205)	0.1416 (0.0407)	0.0132 (0.0069)	0.0123 (0.0159)	0.1573 (0.0497)	0.0132 (0.0075)	0.0139 (0.0171)	0.1591 (0.0506)
6	0.0188 (0.0079)	0.0154 (0.0169)	0.1674 (0.0531)	0.0223 (0.0120)	0.0009 (0.0201)	0.1466 (0.0413)	0.0178 (0.0117)	0.0008 (0.0204)	0.1393 (0.0408)	0.0138 (0.0078)	0.0130 (0.0177)	0.1570 (0.0524)
7	0.0192 (0.0082)	0.0145 (0.0175)	0.1651 (0.0546)	0.0225 (0.0126)	0.0024 (0.0182)	0.1412 (0.0389)	0.0222 (0.0121)	0.0005 (0.0199)	0.1442 (0.0413)	0.0185 (0.0121)	0.0008 (0.0185)	0.1343 (0.0385)
8	0.0222 (0.0122)	0.0025 (0.0176)	0.1386 (0.0388)	0.0235 (0.0128)	0.0025 (0.0176)	0.1386 (0.0388)	0.0226 (0.0126)	0.0025 (0.0176)	0.1386 (0.0388)	0.0226 (0.0124)	0.0025 (0.0176)	0.1386 (0.0388)
9	0.0260 (0.0125)	0.0030 (0.0177)	0.1422 (0.0392)	0.0144 (0.0150)	0.0109 (0.0166)	0.1004 (0.0239)	0.0232 (0.0129)	0.0011 (0.0183)	0.1351 (0.0392)	0.0220 (0.0130)	0.0014 (0.0178)	0.1307 (0.0369)
10	0.0247 (0.0131)	0.0023 (0.0174)	0.1338 (0.0370)	0.0177 (0.0151)	0.0101 (0.0162)	0.1025 (0.0238)	0.0143 (0.0150)	0.0113 (0.0165)	0.0978 (0.0240)	0.0226 (0.0132)	0.0005 (0.0180)	0.1272 (0.0373)

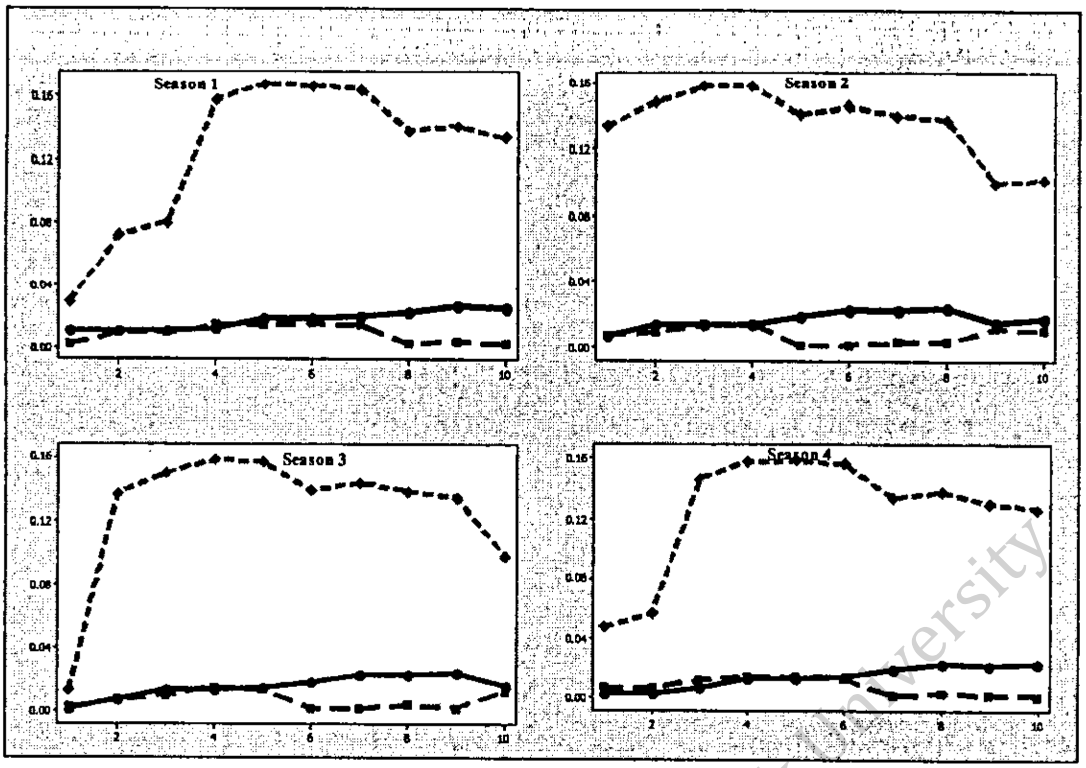

Figure (3.3): The absolute bias for the three estimators for $\text{PAR}_4(1)$ model (Model1) with n=100 and no outlier (—: $\hat{\rho}_k(\nu)$, – – : $\tilde{\rho}_k(\nu)$, ---: $\breve{\rho}_k(\nu)$)

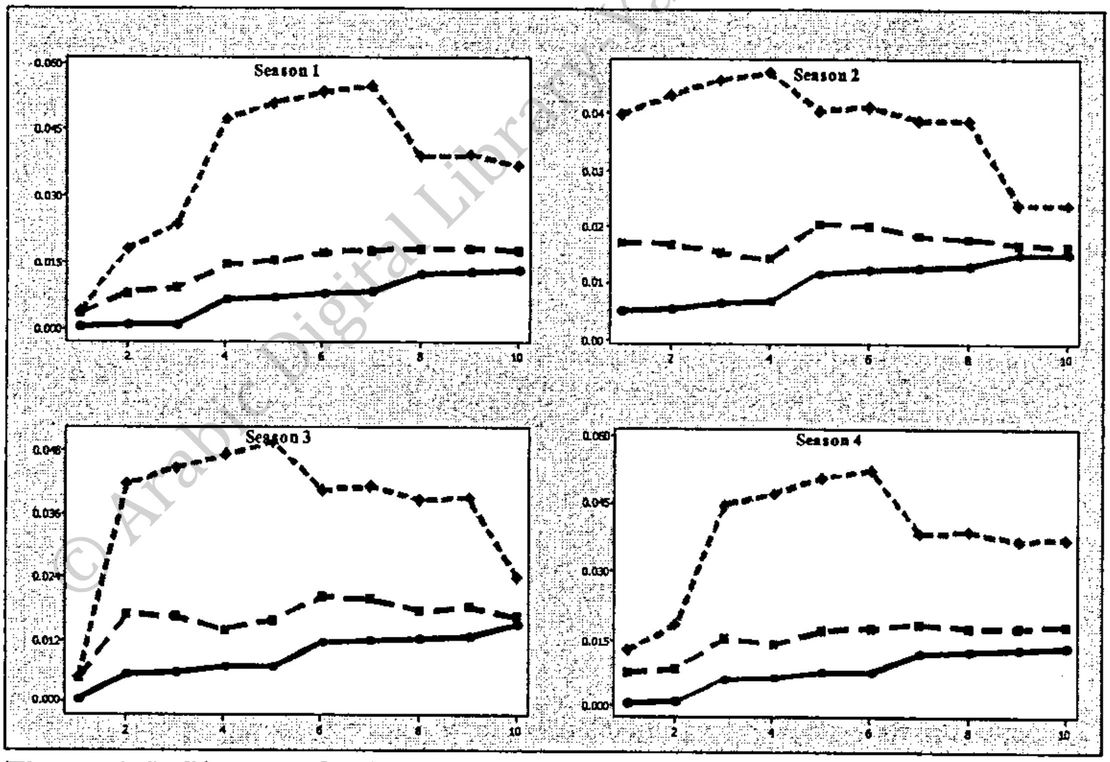

Figure (3.4): The MSE for the three estimators for $\text{PAR}_4(1)$ model (Model 1) with n=100 and no outlier (—: $\hat{\rho}_k(\nu)$, – – : $\tilde{\rho}_k(\nu)$, ---: $\breve{\rho}_k(\nu)$)

Table (3.4): The absolute bias and MSE (in brackets) of the three estimators of the SACF of the $PAR_4(1)$ model (Model 1), n = 30 with additive outlier at season one

	Season											
	1			2			3			4		
Lag	$\hat{\rho}_k$	$\tilde{\rho}_k$	$\breve{\rho}_k$	$\hat{\rho}_k$	$\tilde{\rho}_k$	$\breve{\rho}_k$	$\hat{\rho}_k$	$\tilde{\rho}_k$	$\breve{\rho}_k$	$\hat{\rho}_k$	$\tilde{\rho}_k$	$\breve{\rho}_k$
1	0.6023 (0.3862)	0.0617 (0.0204)	0.1311 (0.0375)	0.3935 (0.1867)	0.0607 (0.0634)	0.1633 (0.0898)	0.0016 (0.0001)	0.0030 (0.0149)	0.0212 (0.0136)	0.0060 (0.0015)	0.0089 (0.0284)	0.0372 (0.0349)
2	0.5544 (0.3332)	0.0697 (0.0355)	0.1509 (0.0675)	0.0508 (0.0222)	0.1004 (0.0630)	0.2239 (0.1048)	0.3870 (0.1814)	0.0606 (0.0617)	0.1690 (0.0918)	0.0074 (0.0022)	0.0117 (0.0290)	0.0540 (0.0450)
3	0.5461 (0.3245)	0.0712 (0.0359)	0.1634 (0.0764)	0.0473 (0.0248)	0.1015 (0.0560)	0.2182 (0.1022)	0.0503 (0.0225)	0.0994 (0.0614)	0.2275 (0.1058)	0.3579 (0.1608)	0.0656 (0.0543)	0.1698 (0.0904)
4	0.4641 (0.2285)	0.1032 (0.0516)	0.2206 (0.1021)	0.0474 (0.0250)	0.1032 (0.0516)	0.2206 (0.1021)	0.0477 (0.0255)	0.1032 (0.0516)	0.2206 (0.1021)	0.0490 (0.0251)	0.1032 (0.0516)	0.2206 (0.1021)
5	0.3592 (0.1616)	0.1293 (0.0600)	0.2560 (0.1207)	0.2470 (0.0988)	0.0539 (0.0542)	0.1537 (0.0766)	0.0485 (0.0258)	0.0983 (0.0537)	0.2183 (0.1045)	0.0489 (0.0289)	0.0914 (0.0571)	0.2004 (0.1052)
6	0.3328 (0.1454)	0.1178 (0.0605)	0.2323 (0.1218)	0.0721 (0.0406)	0.0739 (0.0523)	0.1751 (0.0801)	0.2430 (0.0972)	0.0528 (0.0529)	0.1516 (0.0771)	0.0499 (0.0295)	0.0896 (0.0558)	0.1982 (0.1067)
7	0.3270 (0.1422)	0.1154 (0.0594)	0.2296 (0.1212)	0.0708 (0.0417)	0.0691 (0.0465)	0.1583 (0.0805)	0.0722 (0.0408)	0.0723 (0.0510)	0.1726 (0.0771)	0.2333 (0.0916)	0.0499 (0.0473)	0.1371 (0.0768)
8	0.3307 (0.1235)	0.0693 (0.0440)	0.1561 (0.0795)	0.0712 (0.0416)	0.0693 (0.0440)	0.1561 (0.0795)	0.0713 (0.0418)	0.0693 (0.0440)	0.1561 (0.0795)	0.0728 (0.0416)	0.0693 (0.0440)	0.1561 (0.0795)
9	0.2318 (0.0918)	0.0824 (0.0457)	0.1689 (0.0889)	0.1471 (0.0589)	0.0229 (0.0393)	0.0897 (0.0583)	0.0701 (0.0415)	0.0659 (0.0443)	0.1520 (0.0807)	0.0645 (0.0417)	0.0585 (0.0446)	0.1353 (0.0883)
10	0.2131 (0.0837)	0.0722 (0.0441)	0.1469 (0.1041)	0.0503 (0.0433)	0.0340 (0.0371)	0.0988 (0.0615)	0.1438 (0.0577)	0.0221 (0.0384)	0.0870 (0.0598)	0.0628 (0.0414)	0.0569 (0.0434)	0.1316 (0.0892)

Table (3.5): The absolute bias and MSE (in brackets) of the three estimators of the SACF of the $PAR_4(1)$ model (Model 1), n = 50 with additive outlier at season one

	Season											
	1			2			3			4		
Lag	$\hat{\rho}_k$	$\tilde{\rho}_k$	$\breve{\rho}_k$	$\hat{\rho}_k$	$\tilde{\rho}_k$	$\breve{\rho}_k$	$\hat{\rho}_k$	$\tilde{\rho}_k$	$\breve{\rho}_k$	$\hat{\rho}_k$	$\tilde{\rho}_k$	$\breve{\rho}_k$
1	0.4899 (0.2515)	0.0294 (0.0087)	0.071 (0.0133)	0.3128 (0.1152)	0.0113 (0.0398)	0.1392 (0.0603)	0.0008 (0.0001)	0.0004 (0.0092)	0.012 (0.0087)	0.0029 (0.0008)	0.0024 (0.0173)	0.0481 (0.0237)
2	0.4522 (0.2177)	0.0324 (0.0181)	0.109 (0.0373)	0.0183 (0.0113)	0.0321 (0.0384)	0.1728 (0.0689)	0.3073 (0.113)	0.0141 (0.0382)	0.1414 (0.0619)	0.0037 (0.0012)	0.0058 (0.0188)	0.0562 (0.0317)
3	0.4444 (0.2106)	0.035 (0.0199)	0.1149 (0.045)	0.0218 (0.0139)	0.0352 (0.033)	0.1779 (0.07)	0.0186 (0.012)	0.0343 (0.037)	0.1738 (0.0701)	0.2816 (0.098)	0.0187 (0.0329)	0.1495 (0.0641)
4	0.4148 (0.1825)	0.0389 (0.03)	0.1776 (0.0708)	0.0223 (0.0142)	0.0389 (0.03)	0.1776 (0.0708)	0.0218 (0.0144)	0.0389 (0.03)	0.1776 (0.0708)	0.0181 (0.0136)	0.0389 (0.03)	0.1776 (0.0708)
5	0.2916 (0.1043)	0.0538 (0.033)	0.2001 (0.0794)	0.1939 (0.0589)	0.0023 (0.0429)	0.137 (0.0579)	0.0226 (0.0149)	0.0375 (0.0326)	0.1738 (0.074)	0.0227 (0.0164)	0.0329 (0.0341)	0.1702 (0.0743)
6	0.2705 (0.0929)	0.0483 (0.0344)	0.1903 (0.0803)	0.032 (0.0261)	0.014 (0.0411)	0.1497 (0.0594)	0.1902 (0.0574)	0.0034 (0.0417)	0.1339 (0.0588)	0.0236 (0.0168)	0.0332 (0.0346)	0.1662 (0.0769)
7	0.2664 (0.0909)	0.0483 (0.0349)	0.1859 (0.082)	0.0352 (0.0273)	0.0138 (0.0363)	0.1405 (0.0559)	0.0332 (0.0265)	0.0148 (0.04)	0.1463 (0.0597)	0.1767 (0.0529)	0.004 (0.0371)	0.1259 (0.056)
8	0.2677 (0.082)	0.0155 (0.0346)	0.137 (0.0562)	0.0361 (0.0278)	0.0155 (0.0346)	0.137 (0.0562)	0.0362 (0.0276)	0.0155 (0.0346)	0.137 (0.0562)	0.0356 (0.0283)	0.0155 (0.0346)	0.137 (0.0562)
9	0.1711 (0.0504)	0.0239 (0.035)	0.146 (0.0573)	0.1123 (0.0354)	0.0169 (0.0361)	0.0861 (0.0388)	0.0359 (0.0281)	0.0143 (0.0355)	0.132 (0.0582)	0.0341 90.0293)	0.0106 (0.0343)	0.1244 (0.0546)
10	0.1561 (0.0458)	0.0194 (0.0335)	0.1328 (0.0549)	0.0259 (0.0312)	0.0099 (0.0348)	0.0918 (0.0381)	0.1095 (0.0343)	0.0163 (0.0354)	0.0827 (0.0406)	0.0342 (0.0299)	0.0104 (0.0344)	0.1196 (0.0566)

Table (3.6): The absolute bias and MSE (in brackets) of the three estimators of the SACF of the $PAR_4(1)$ model (Model 1), n = 100 with additive outlier at season one

	Season											
	1			2			3			4		
Lag	$\hat{\rho}_k$	$\tilde{\rho}_k$	$\breve{\rho}_k$	$\hat{\rho}_k$	$\tilde{\rho}_k$	$\breve{\rho}_k$	$\hat{\rho}_k$	$\tilde{\rho}_k$	$\breve{\rho}_k$	$\hat{\rho}_k$	$\tilde{\rho}_k$	$\breve{\rho}_k$
1	0.3583 (0.132)	0.0144 (0.0035)	0.0425 (0.0055)	0.2302 (0.0614)	0.0135 (0.0175)	0.1393 (0.0412)	0.0004 (0)	0.0013 (0.0045)	0.0121 (0.0049)	0.0019 (0.0004)	0.0061 (0.0078)	0.0477 (0.0129)
2	0.3293 (0.1131)	0.0206 (0.0083)	0.0841 (0.0203)	0.0131 (0.0053)	0.0237 (0.017)	0.1596 (0.0466)	0.2255 (0.0594)	0.0133 (0.0175)	0.1416 (0.0429)	0.002 (0.0006)	0.0063 (0.0086)	0.0564 (0.0184)
3	0.3228 (0.1091)	0.0204 (0.009)	0.0914 (0.0257)	0.0134 (0.0064)	0.027 (0.0157)	0.1678 (0.0489)	0.013 (0.0056)	0.0233 (0.0169)	0.1611 (0.0481)	0.2078 (0.0523)	0.0175 (0.0159)	0.1513 (0.0454)
4	0.3361 (0.1197)	0.0274 (0.0146)	0.1682 (0.0499)	0.0135 (0.0066)	0.0274 (0.0146)	0.1682 (0.0499)	0.0129 (0.0067)	0.0274 (0.0146)	0.1682 (0.0499)	0.0126 (0.0065)	0.0274 (0.0146)	0.1682 (0.0499)
5	0.2133 (0.0536)	0.0351 (0.0156)	0.1825 (0.0554)	0.1433 (0.0317)	0.012 (0.0201)	0.1484 (0.0417)	0.0132 (0.0069)	0.0253 (0.0163)	0.1664 (0.0522)	0.0132 (0.0075)	0.026 (0.0175)	0.1675 (0.0524)
6	0.196 (0.0469)	0.0338 (0.0174)	0.18 (0.0566)	0.0223 (0.012)	0.0178 (0.0194)	0.1559 (0.0431)	0.1406 (0.0311)	0.0112 (0.02)	0.146 (0.0416)	0.0138 (0.0078)	0.0248 (0.018)	0.1652 (0.054)
7	0.1927 (0.0458)	0.0325 (0.0178)	0.1774 (0.0578)	0.0225 (0.0126)	0.0177 (0.0177)	0.1497 (0.04)	0.0222 (0.0121)	0.017 (0.0192)	0.1533 (0.0428)	0.1298 (0.0281)	0.0118 (0.0181)	0.1405 (0.0388)
8	0.2092 (0.0518)	0.0175 (0.0171)	0.1469 (0.0397)	0.0235 (0.0128)	0.0175 (0.0171)	0.1469 (0.0397)	0.0226 (0.0126)	0.0175 (0.0171)	0.1469 (0.0397)	0.0226 (0.0124)	0.0175 (0.0171)	0.1469 (0.0397)
9	0.1309 (0.0288)	0.0219 (0.017)	0.1523 (0.0408)	0.0882 (0.0194)	0.0012 (0.0154)	0.1062 (0.0235)	0.0232 (0.0129)	0.0158 (0.0177)	0.1433 (0.0398)	0.022 (0.013)	0.0151 (0.0172)	0.1384 (0.0371)
10	0.1219 (0.0265)	0.0196 (0.0167)	0.1432 (0.0378)	0.0177 (0.0151)	0.0047 (0.0147)	0.1093 (0.0236)	0.0866 (0.0192)	0.0006 (0.0152)	0.1035 (0.0233)	0.0226 (0.0132)	0.014 (0.0173)	0.1348 (0.0372)

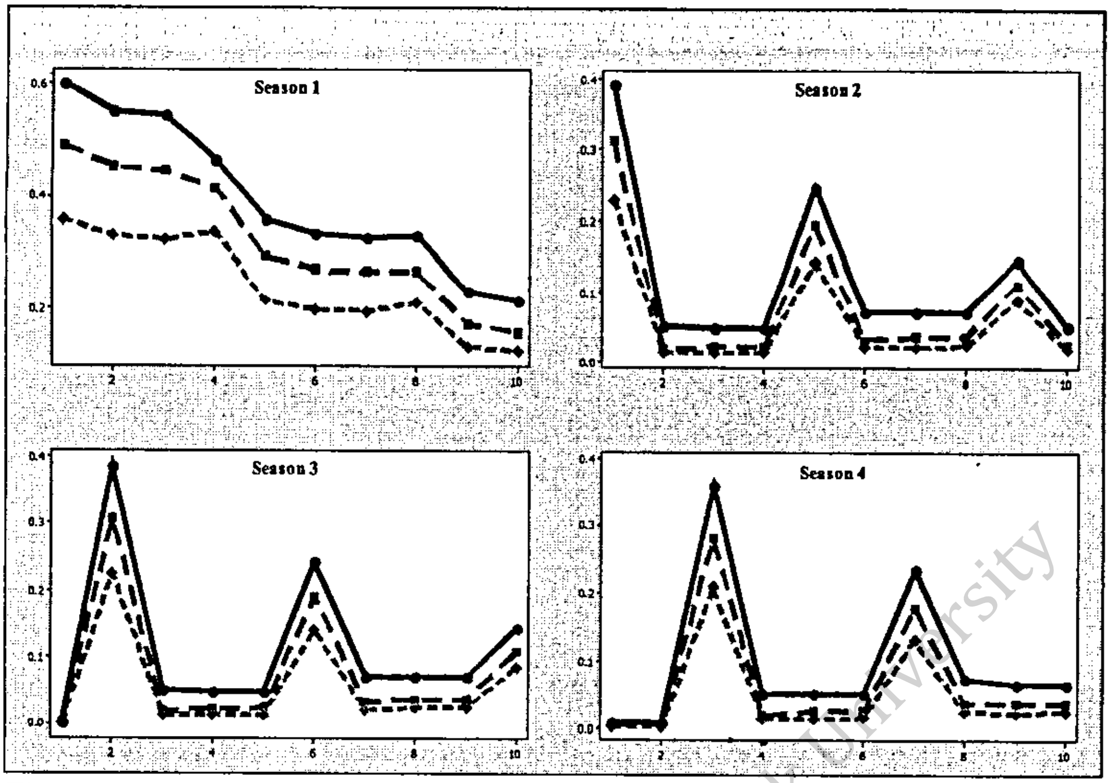

Figure (3.5): The absolute bias for $\hat{\rho}_k(\nu)$ for $PAR_4(1)$ model (Model 1) with a single additive outlier at season one (—: n=30, — — : n=50, ---: n=100)

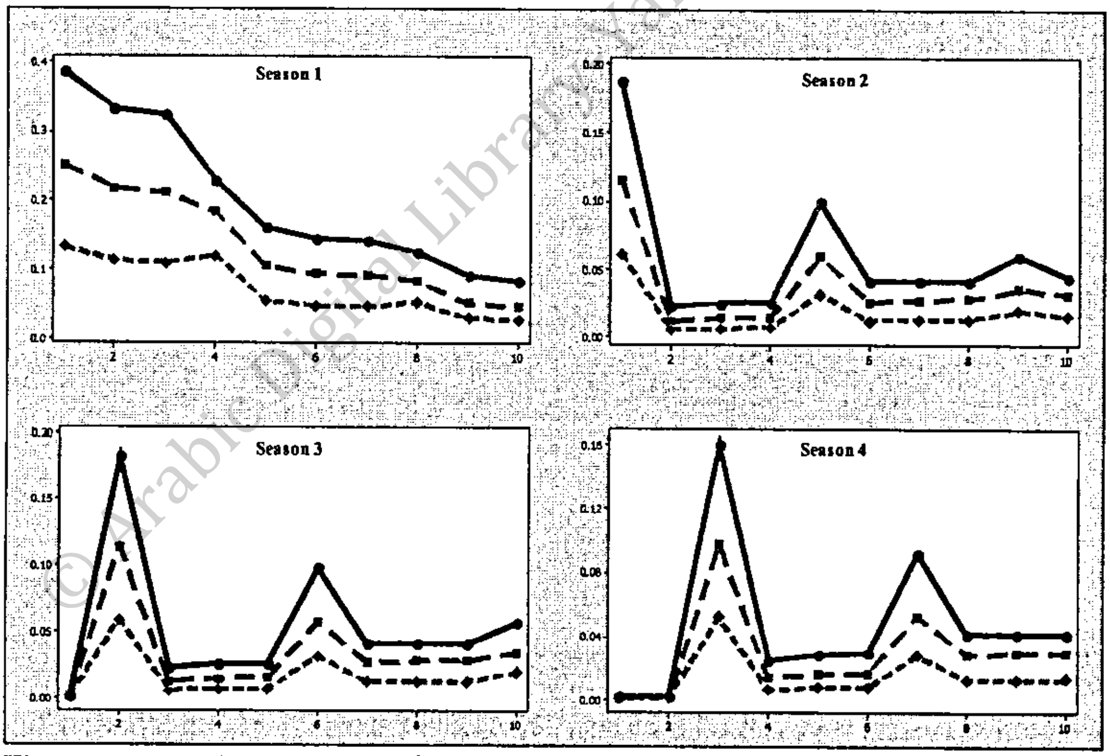

Figure (3.6): The MSE for $\hat{\rho}_k(\nu)$ for $PAR_4(1)$ model (Model 1) with a single additive outlier at season one (—: n=30, — — : n=50, ---: n=100)

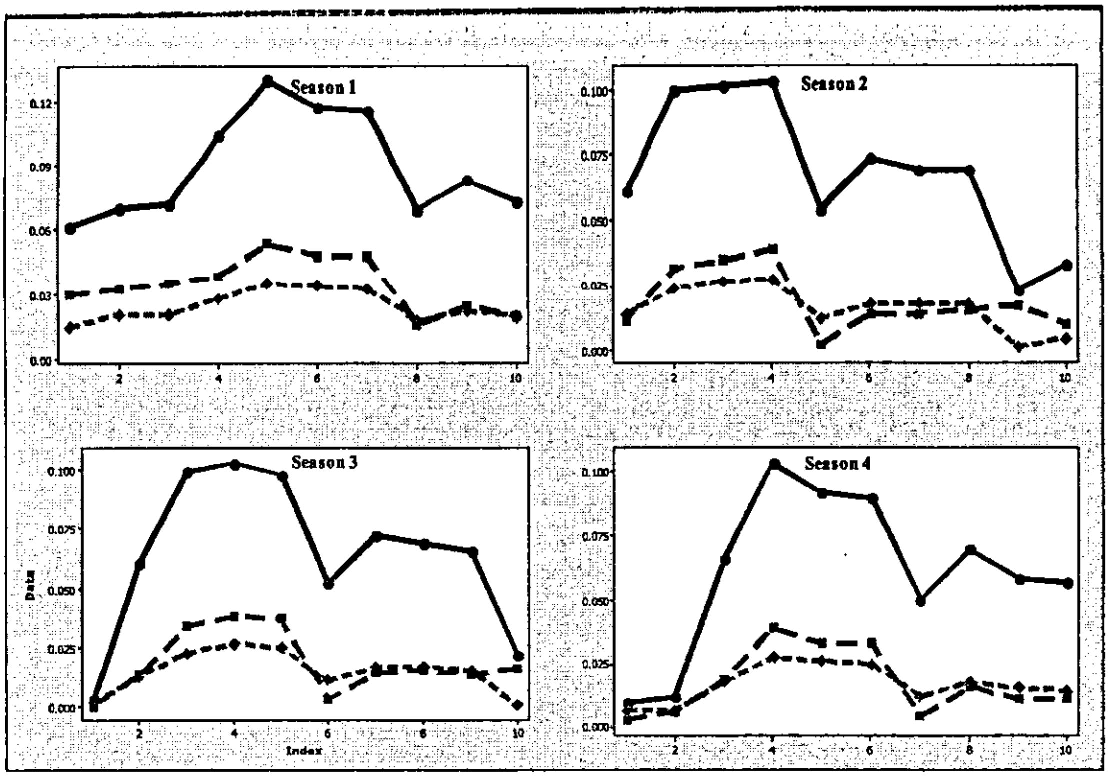

Figure (3.7): The absolute bias for $\tilde{\rho}_k(\nu)$ for $PAR_4(1)$ model (Model 1) with a single additive outlier at season one (—: n=30, – – : n=50, ---: n=100)

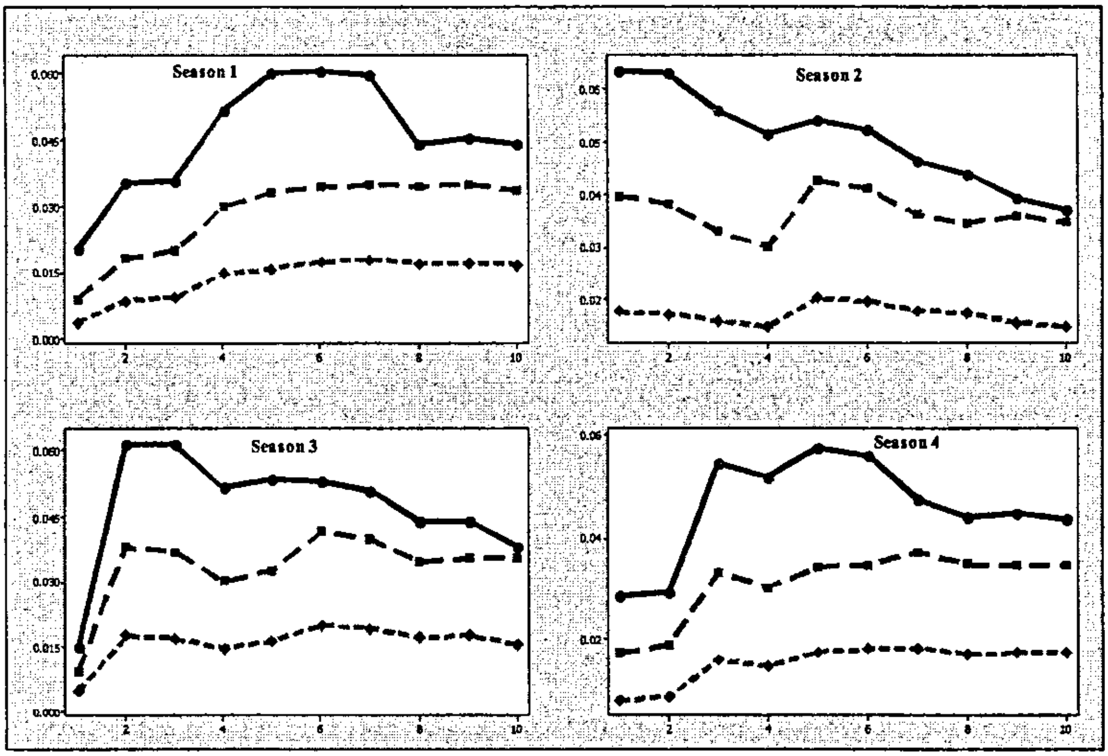

Figure (3.8): The MSE for $\tilde{\rho}_k(\nu)$ for $PAR_4(1)$ model (Model 1) with a single additive outlier at season one (—: n=30, – – : n=50, ---: n=100)

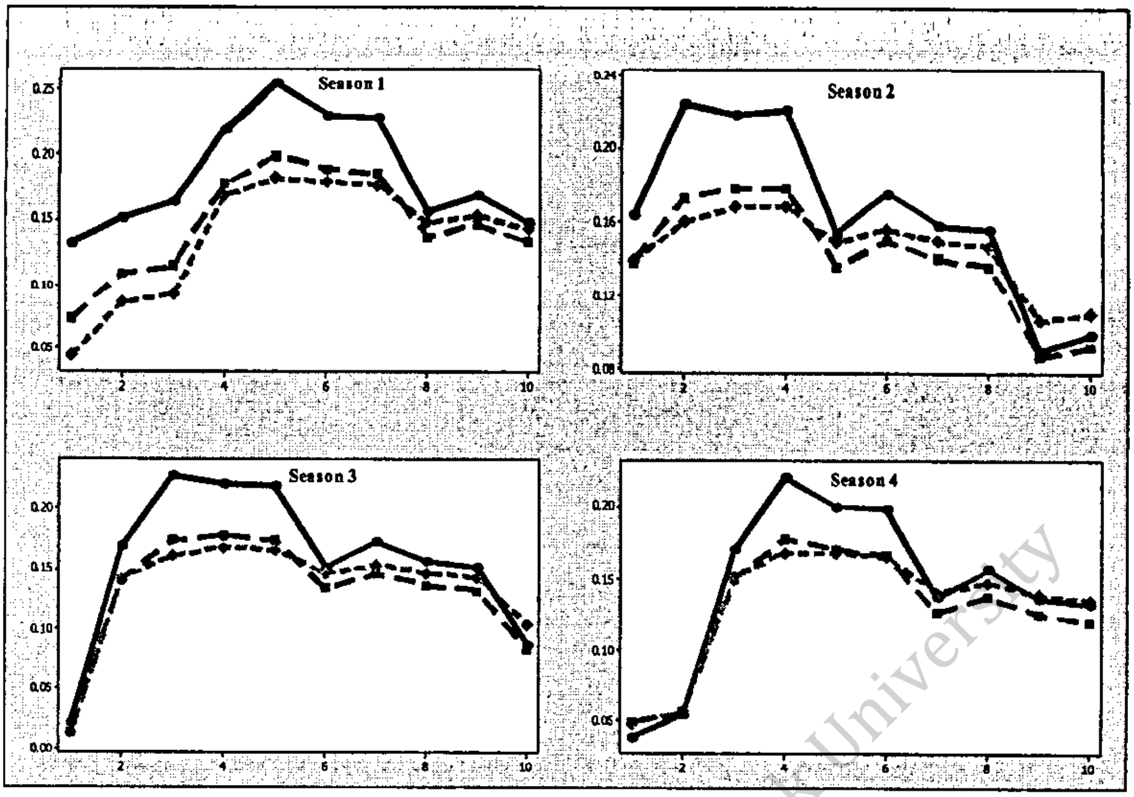

Figure (3.9): The absolute bias for $\breve{\rho}_k(\nu)$ for $PAR_4(1)$ model (Model 1) with a single additive outlier at season one (—: n=30, – – : n=50, ----: n=100)

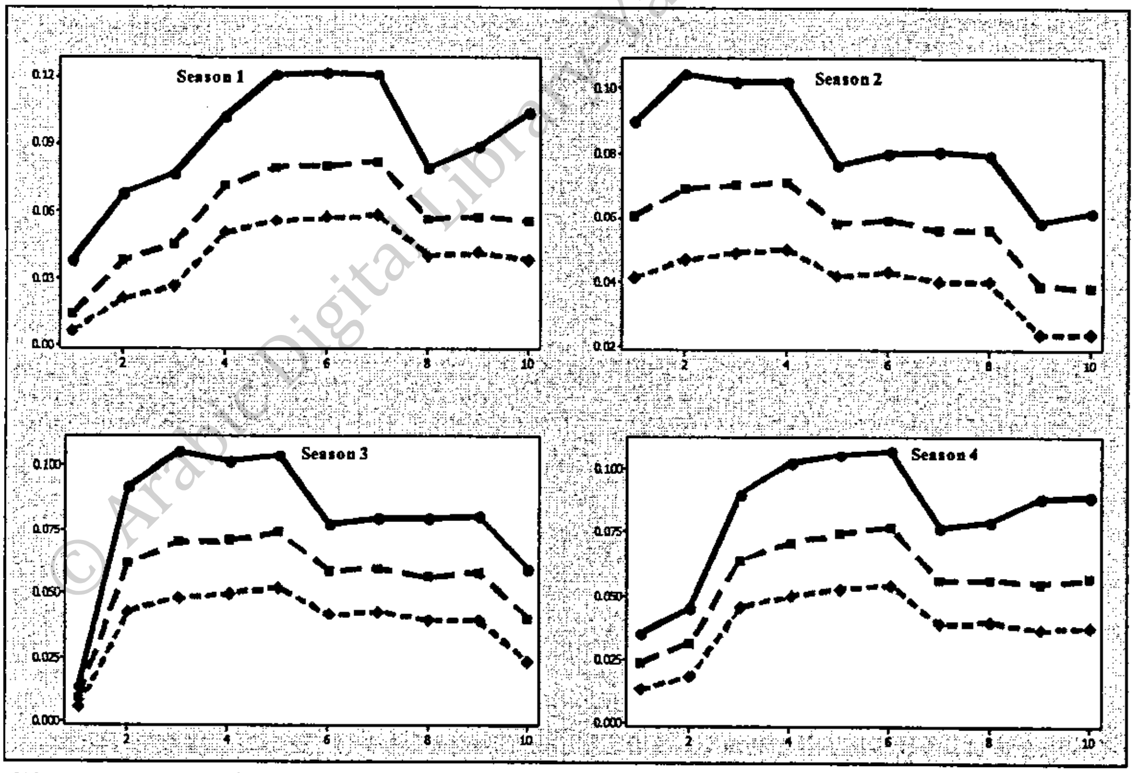

Figure (3.10): The MSE for $\breve{\rho}_k(\nu)$ for $PAR_4(1)$ model (Model 1) with a single additive outlier at season one (—: n=30, – – : n=50, ----: n=100)

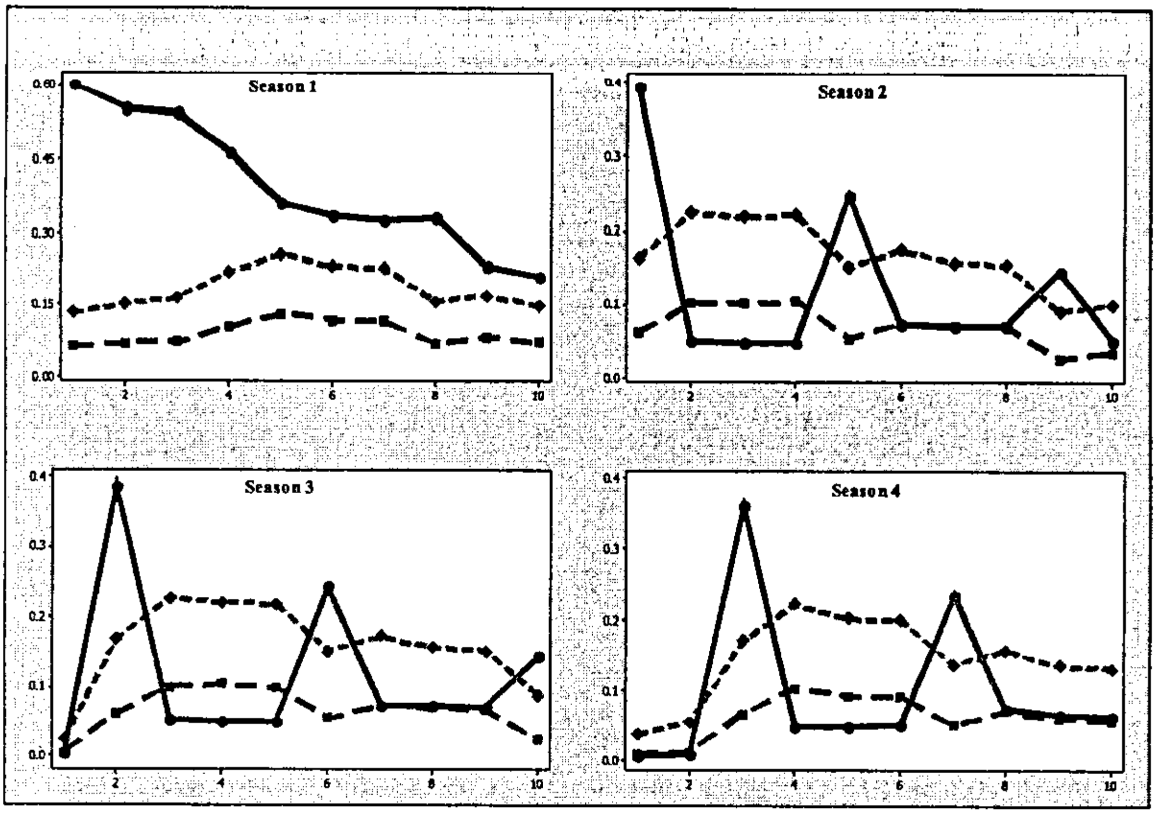

Figure (3.11): The absolute bias for the three estimators for $PAR_4(1)$ model (Model 1) with a single additive outlier at season one and n=30 (—: $\hat{\rho}_k(\nu)$, – – : $\tilde{\rho}_k(\nu)$, ---: $\breve{\rho}_k(\nu)$)

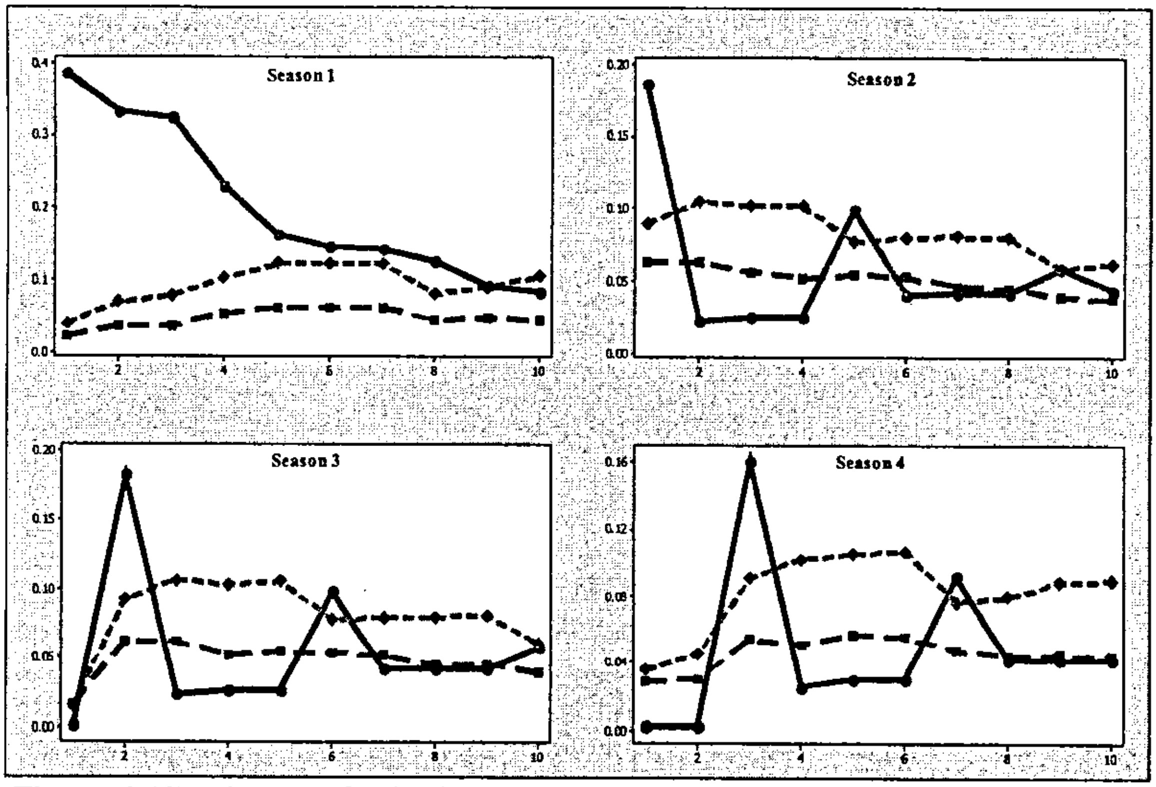

Figure (3.12): The MSE for the three estimators for $PAR_4(1)$ model (Model 1) with a single additive outlier at season one and n=30 (—: $\hat{\rho}_k(\nu)$, – – : $\tilde{\rho}_k(\nu)$, ---: $\breve{\rho}_k(\nu)$)

Table (3.7): The absolute bias and MSE (in brackets) of the moment estimator of the SACF ($\hat{\rho}_k(\nu)$) of the $PAR_{12}(1)$ model (Model 2), n = 100 with additive outlier at season one

	Season											
lag	1	2	3	4	5	6	7	8	9	10	11	12
1	0.3253 (0.1082)	0.2152 (0.0520)	0.0001 (0.0000)	0.0004 (0.0002)	0.0007 (0.0001)	0.0027 (0.0031)	0.0021 (0.0017)	0.0036 (0.0062)	0.0012 (0.0078)	0.0010 (0.0022)	0.0014 (0.0002)	0.0013 (0.0027)
2	0.2274 (0.0575)	0.0066 (0.0031)	0.2122 (0.0508)	0.0006 (0.0003)	0.0005 (0.0005)	0.0047 (0.0034)	0.0009 (0.0050)	0.0038 (0.0079)	0.0005 (0.0098)	0.0016 (0.0082)	0.0023 (0.0029)	0.0024 (0.0035)
3	0.2105 (0.0506)	0.0067 (0.0061)	0.0070 (0.0033)	0.1980 (0.0453)	0.0008 (0.0007)	0.0051 (0.0042)	0.0035 (0.0055)	0.0005 (0.0091)	0.0000 (0.0100)	0.0031 (0.0094)	0.0004 (0.0085)	0.0030 (0.0061)
4	0.1589 (0.0336)	0.0071 (0.0067)	0.0068 (0.0061)	0.0065 (0.0039)	0.1889 (0.0423)	0.0053 (0.0045)	0.0039 (0.0059)	0.0011 (0.0090)	0.0024 (0.0104)	0.0030 (0.0098)	0.0012 (0.0093)	0.0031 (0.0092)
5	0.0446 (0.0120)	0.0088 (0.0080)	0.0068 (0.0067)	0.0062 (0.0064)	0.0060 (0.0044)	0.1311 (0.0260)	0.0042 (0.0060)	0.0013 (0.0090)	0.0043 (0.0102)	0.0032 (0.0096)	0.0025 (0.0095)	0.0000 (0.0098)
6	0.0214 (0.0105)	0.0037 (0.0099)	0.0082 (0.0081)	0.0063 (0.0070)	0.0056 (0.0067)	0.0060 (0.0067)	0.1019 (0.0197)	0.0021 (0.0090)	0.0052 (0.0102)	0.0048 (0.0096)	0.0023 (0.0092)	0.0013 (0.0104)
7	0.0182 (0.0103)	0.0072 (0.0100)	0.0043 (0.0098)	0.0098 (0.0085)	0.0060 (0.0073)	0.0033 (0.0079)	0.0049 (0.0077)	0.0454 (0.0116)	0.0056 (0.0102)	0.0049 (0.0098)	0.0046 (0.0093)	0.0078 (0.0104)
8	0.0061 (0.0103)	0.0094 (0.0104)	0.0077 (0.0100)	0.0018 (0.0097)	0.0097 (0.0085)	0.0037 (0.0082)	0.0039 (0.0088)	0.0014 (0.0100)	0.0120 (0.0104)	0.0054 (0.0097)	0.0046 (0.0092)	0.0100 (0.0105)
9	0.0039 (0.0103)	0.0148 (0.0104)	0.0097 (0.0104)	0.0069 (0.0097)	0.0013 (0.0100)	0.0060 (0.0088)	0.0041 (0.0093)	0.0018 (0.0097)	0.0075 (0.0108)	0.0081 (0.0098)	0.0050 (0.0091)	0.0114 (0.0105)
10	0.0019 (0.0102)	0.0166 (0.0103)	0.0151 (0.0103)	0.0088 (0.0104)	0.0082 (0.0100)	0.0046 (0.0100)	0.0042 (0.0095)	0.0032 (0.0099)	0.0079 (0.0102)	0.0076 (0.0103)	0.0078 (0.0094)	0.0117 (0.0102)

Table (3.8): The absolute bias and MSE (in brackets) of the second estimator of the SACF ($\tilde{\rho}_k(\nu)$) of the $PAR_{12}(1)$ model (Model 2), n = 100 with additive outlier at season one

	Season											
lag	1	2	3	4	5	6	7	8	9	10	11	12
1	0.0122 (0.0033)	0.0094 (0.0148)	0.0017 (0.0044)	0.0025 (0.0076)	0.0022 (0.0064)	0.0083 (0.0136)	0.0045 (0.0120)	0.0075 (0.0193)	0.0125 (0.0210)	0.0107 (0.0123)	0.0020 (0.0086)	0.0047 (0.0138)
2	0.0141 (0.0140)	0.0187 (0.0144)	0.0114 (0.0149)	0.0005 (0.0087)	0.0014 (0.0095)	0.0108 (0.0133)	0.0098 (0.0131)	0.0080 (0.0136)	0.0080 (0.0065)	0.0126 (0.0126)	0.0130 (0.0128)	0.0070 (0.0143)
3	0.0157 (0.0142)	0.0175 (0.0121)	0.0205 (0.0144)	0.0101 (0.0148)	0.0043 (0.0101)	0.0100 (0.0128)	0.0116 (0.0125)	0.0082 (0.0079)	0.0067 (0.0041)	0.0074 (0.0038)	0.0122 (0.0110)	0.0131 (0.0119)
4	0.0196 (0.0117)	0.0178 (0.0115)	0.0186 (0.0122)	0.0186 (0.0142)	0.0118 (0.0148)	0.0117 (0.0127)	0.0106 (0.0117)	0.0085 (0.0074)	0.0053 (0.0021)	0.0062 (0.0024)	0.0070 (0.0033)	0.0097 (0.0063)
5	0.0114 (0.0061)	0.0182 (0.0084)	0.0186 (0.0116)	0.0167 (0.0114)	0.0198 (0.0143)	0.0135 (0.0108)	0.0119 (0.0114)	0.0082 (0.0064)	0.0053 (0.0019)	0.0047 (0.0012)	0.0058 (0.0021)	0.0054 (0.0018)
6	0.0062 (0.0018)	0.0095 (0.0031)	0.0188 (0.0083)	0.0167 (0.0108)	0.0173 (0.0112)	0.0187 (0.0105)	0.0125 (0.0080)	0.0085 (0.0063)	0.0050 (0.0017)	0.0046 (0.0011)	0.0044 (0.0010)	0.0045 (0.0011)
7	0.0052 (0.0011)	0.0050 (0.0009)	0.0097 (0.0029)	0.0169 (0.0076)	0.0174 (0.0104)	0.0150 (0.0074)	0.0165 (0.0078)	0.0075 (0.0036)	0.0051 (0.0016)	0.0043 (0.0010)	0.0044 (0.0010)	0.0033 (0.0006)
8	0.0038 (0.0006)	0.0041 (0.0005)	0.0050 (0.0008)	0.0088 (0.0026)	0.0173 (0.0072)	0.0149 (0.0066)	0.0133 (0.0051)	0.0095 (0.0034)	0.0041 (0.0008)	0.0044 (0.0009)	0.0041 (0.0008)	0.0033 (0.0005)
9	0.0037 (0.0005)	0.0029 (0.0003)	0.0041 (0.0005)	0.0046 (0.0007)	0.0088 (0.0024)	0.0137 (0.0044)	0.0131 (0.0046)	0.0074 (0.0020)	0.0047 (0.0008)	0.0034 (0.0005)	0.0042 (0.0008)	0.0030 (0.0005)
10	0.0034 (0.0005)	0.0028 (0.0003)	0.0029 (0.0003)	0.0038 (0.0005)	0.0045 (0.0007)	0.0065 (0.0012)	0.0118 (0.0029)	0.0071 (0.0018)	0.0035 (0.0004)	0.0039 (0.0004)	0.0032 (0.0004)	0.0031 (0.0005)

Table (3.9): The absolute bias and MSE (in brackets) of the third estimator of the SACF ($\breve{\rho}_k(\nu)$) of the $PAR_{12}(1)$ model (Model 2), n = 100 with additive outlier at season one

	Season											
lag	1	2	3	4	5	6	7	8	9	10	11	12
1	0.0381 (0.0049)	0.1352 (0.0380)	0.0140 (0.0048)	0.0469 (0.0126)	0.0285 (0.0081)	0.1382 (0.0376)	0.1051 (0.0278)	0.1437 (0.0380)	0.1227 (0.0286)	0.1152 (0.0297)	0.0511 (0.0140)	0.1242 (0.0341)
2	0.1456 (0.0396)	0.1548 (0.0435)	0.1406 (0.0396)	0.0580 (0.0174)	0.0687 (0.0212)	0.1462 (0.0410)	0.1607 (0.0421)	0.1425 (0.0311)	0.0836 (0.0095)	0.1116 (0.0194)	0.1371 (0.0372)	0.1413 (0.0411)
3	0.1598 (0.0461)	0.1726 (0.0429)	0.1597 (0.0448)	0.1536 (0.0434)	0.0785 (0.0253)	0.1574 (0.0442)	0.1632 (0.0423)	0.1248 (0.0205)	0.0710 (0.0063)	0.0689 (0.0059)	0.1089 (0.0176)	0.1615 (0.0400)
4	0.1728 (0.0432)	0.1729 (0.0425)	0.1739 (0.0431)	0.1701 (0.0481)	0.1589 (0.0447)	0.1601 (0.0457)	0.1661 (0.0418)	0.1223 (0.0194)	0.0539 (0.0034)	0.0573 (0.0039)	0.0655 (0.0052)	0.0899 (0.0105)
5	0.0919 (0.0107)	0.1582 (0.0318)	0.1737 (0.0424)	0.1741 (0.0420)	0.1741 (0.0490)	0.1656 (0.0382)	0.1669 (0.0420)	0.1183 (0.0177)	0.0521 (0.0031)	0.0425 (0.0020)	0.0542 (0.0034)	0.0506 (0.0029)
6	0.0511 (0.0030)	0.0712 (0.0059)	0.1579 (0.0314)	0.1723 (0.0405)	0.1729 (0.0409)	0.1735 (0.0403)	0.1519 (0.0297)	0.1174 (0.0173)	0.0493 (0.0028)	0.0410 (0.0019)	0.0399 (0.0018)	0.0412 (0.0019)
7	0.0415 (0.0019)	0.0377 (0.0015)	0.0704 (0.0057)	0.1533 (0.0292)	0.1704 (0.0391)	0.1495 (0.0275)	0.1570 (0.0308)	0.0927 (0.0099)	0.0486 (0.0027)	0.0386 (0.0016)	0.0385 (0.0016)	0.0297 (0.0010)
8	0.0299 (0.0010)	0.0302 (0.0010)	0.0372 (0.0015)	0.0666 (0.0051)	0.1500 (0.0277)	0.1436 (0.0253)	0.1291 (0.0195)	0.0941 (0.0100)	0.0358 (0.0014)	0.0381 (0.0016)	0.0362 (0.0014)	0.0286 (0.0009)
9	0.0287 (0.0009)	0.0214 (0.0005)	0.0298 (0.0009)	0.0349 (0.0013)	0.0643 (0.0047)	0.1197 (0.0165)	0.1228 (0.0177)	0.0724 (0.0057)	0.0360 (0.0014)	0.0277 (0.0008)	0.0357 (0.0014)	0.0268 (0.0008)
10	0.0269 (0.0008)	0.0206 (0.0004)	0.0211 (0.0005)	0.0279 (0.0008)	0.0336 (0.0012)	0.0472 (0.0024)	0.1000 (0.0112)	0.0680 (0.0051)	0.0266 (0.0007)	0.0278 (0.0008)	0.0258 (0.0007)	0.0264 (0.0007)

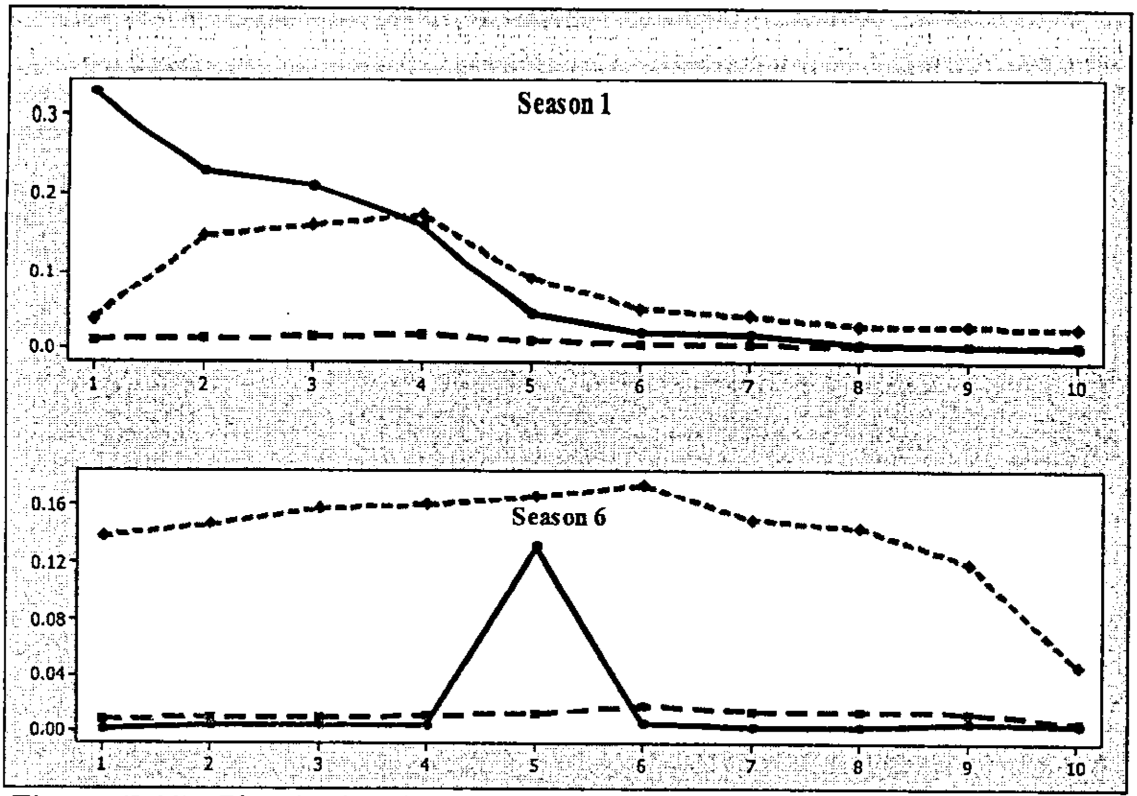

Figure (3.13): The absolute bias for specific seasons for the three estimators for $PAR_{12}(1)$ model (Model 2) with a single additive outlier at season one and n=100 (—: $\hat{\rho}_k(\nu)$, — —: $\tilde{\rho}_k(\nu)$, ----: $\breve{\rho}_k(\nu)$)

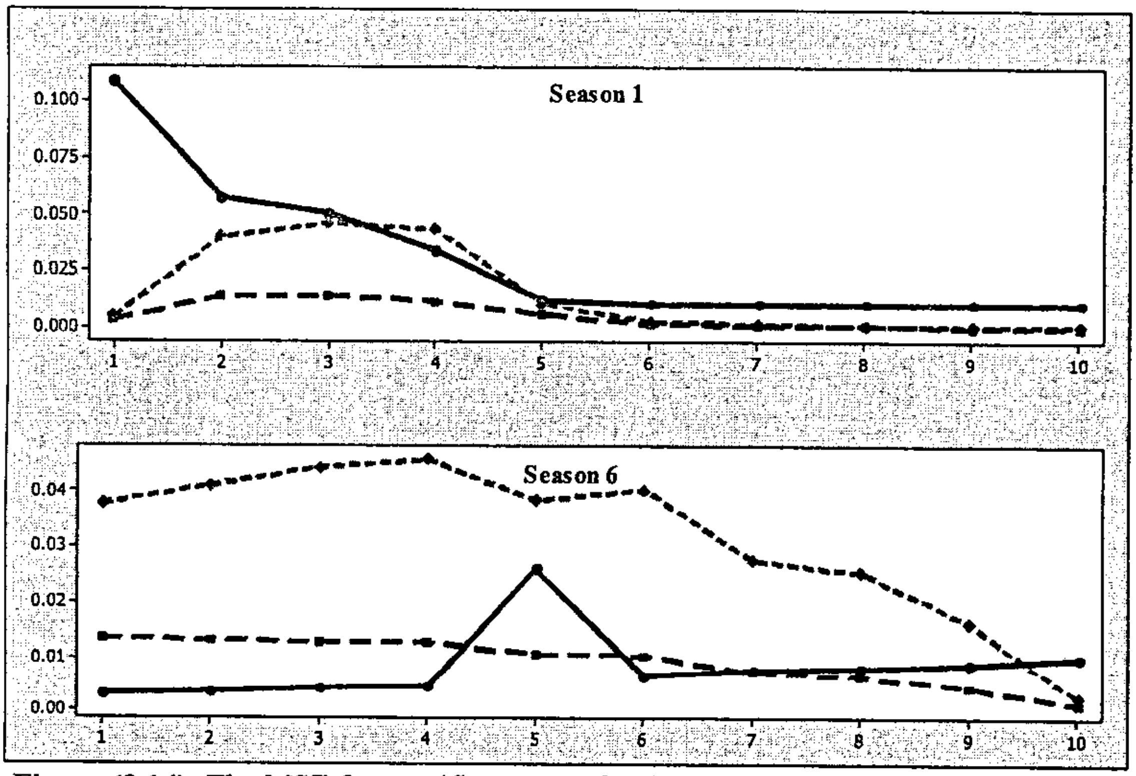

Figure (3.14): The MSE for specific seasons for the three estimators for $PAR_{12}(1)$ model (Model 2) with a single additive outlier at season one and n=100 (—: $\hat{\rho}_k(\nu)$, — —: $\tilde{\rho}_k(\nu)$, ----: $\breve{\rho}_k(\nu)$)

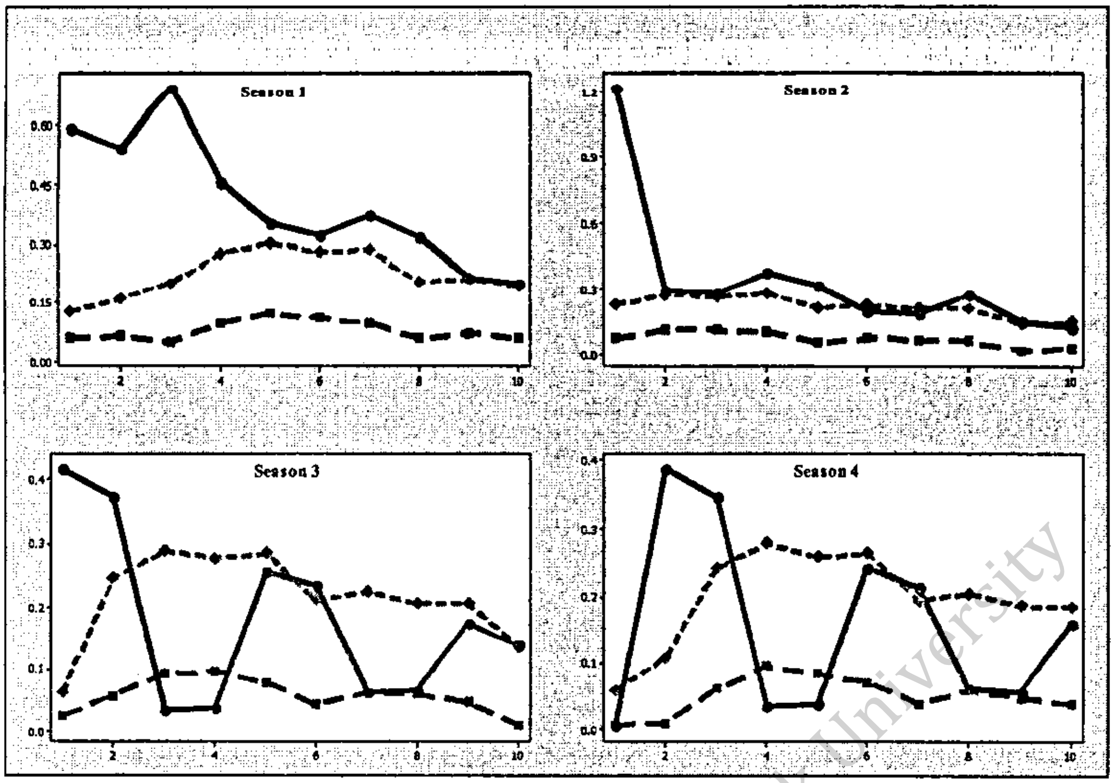

Figure (3.15): The absolute bias for the three estimators for $PAR_4(1)$ model (Model 1) with two additive outliers at seasons one and two and n=30 (—: $\hat{\rho}_k(\nu)$, – –: $\tilde{\rho}_k(\nu)$, ----: $\breve{\rho}_k(\nu)$)

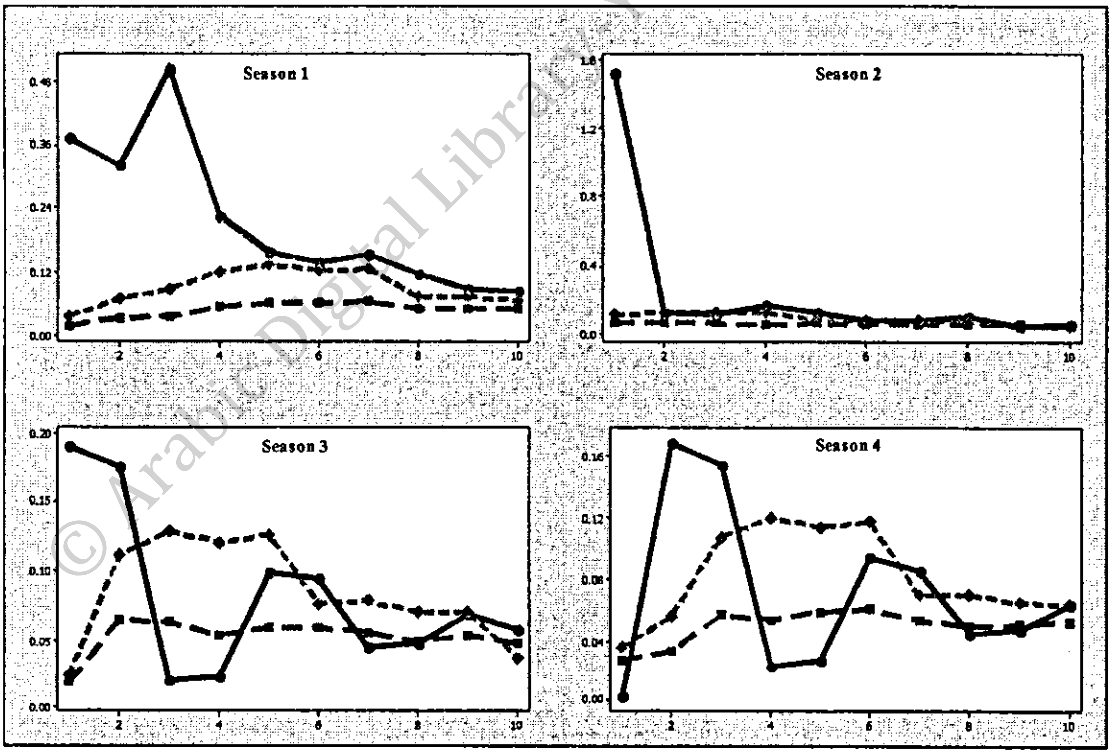

Figure (3.16): The MSE for the three estimators for $PAR_4(1)$ model (Model 1) with two additive outliers at seasons one and two and n=30 (—: $\hat{\rho}_k(\nu)$, – –: $\tilde{\rho}_k(\nu)$, ----: $\breve{\rho}_k(\nu)$)

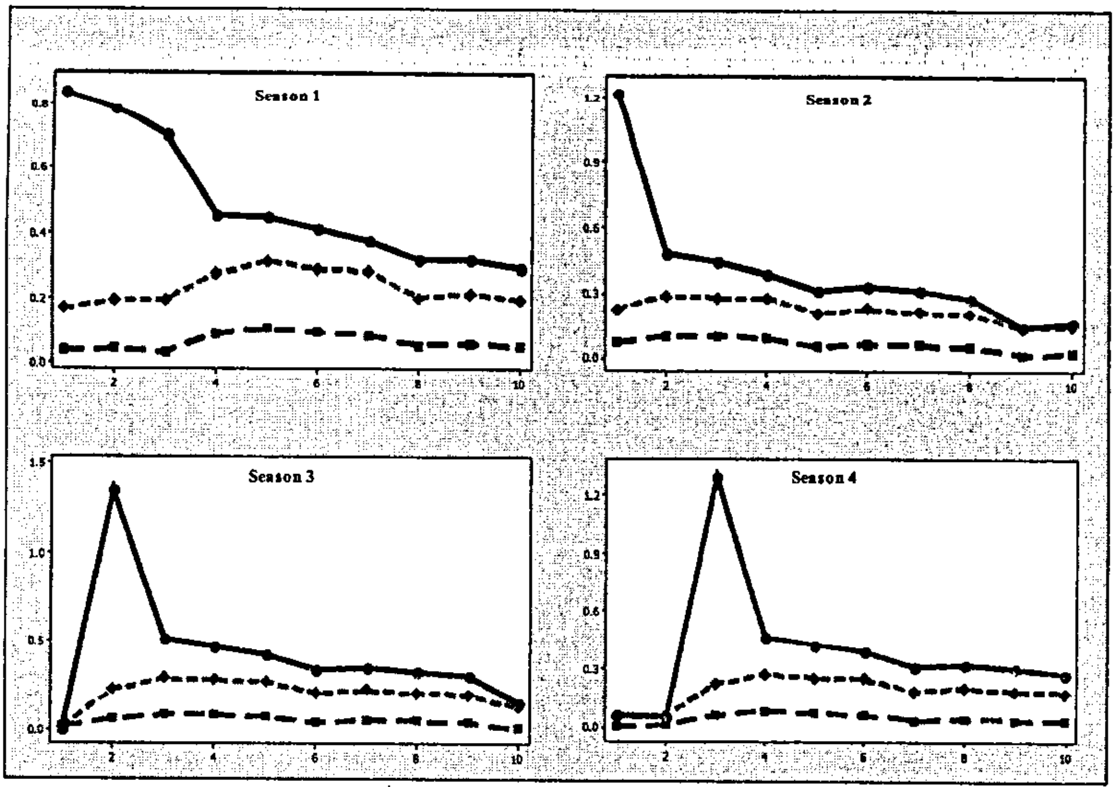

Figure (3.17): The absolute bias for the three estimators for $PAR_4(1)$ model (Model 1) with four additive outliers at seasons one, two, three and four and n=30 (——: $\hat{\rho}_k(\nu)$, — — : $\tilde{\rho}_k(\nu)$, ----: $\breve{\rho}_k(\nu)$)

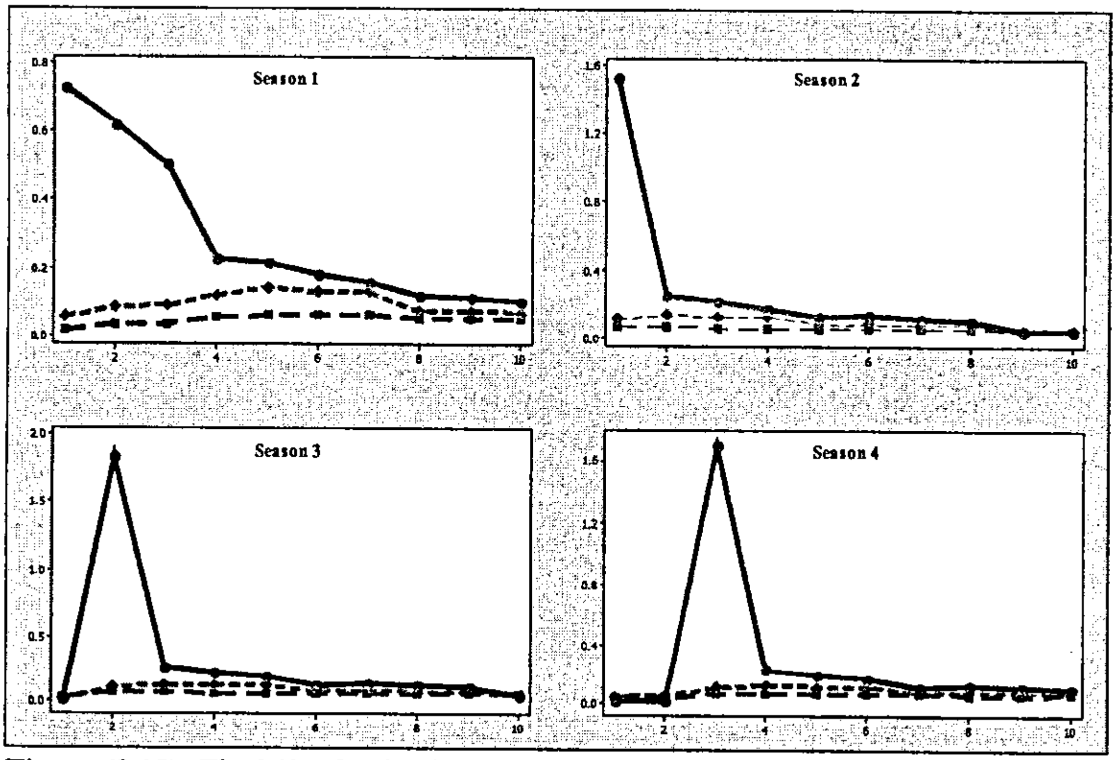

Figure (3.18): The MSE for the three estimators for $PAR_4(1)$ model (Model 1) with four additive outliers at seasons one, two, three and four and n=30 (——: $\hat{\rho}_k(\nu)$, — — : $\tilde{\rho}_k(\nu)$, ----: $\breve{\rho}_k(\nu)$)

3.4 Results

From Tables (3.3)-(3.5) and Figures (3.5)-(3.10) we can notice that the absolute bias of the three estimators are decreases as n increases. Also, all the estimators are seem to be consistent sense the MSE goes to zero as n increases. We can see from Figures (3.3) and (3.4) that if there is no outliers in the data, then the MSE of the moment estimator $\hat{\rho}_k(\nu)$ is less than those for the estimators $\tilde{\rho}_k(\nu)$ and $\breve{\rho}_k(\nu)$ and in almost all cases the absolute bias of $\hat{\rho}_k(\nu)$ is less than that of $\tilde{\rho}_k(\nu)$ and $\breve{\rho}_k(\nu)$.

In the case of a single additive outlier, the moment estimator $\hat{\rho}_k(\nu)$ has a greater bias and MSE in season one while this is not true for the others seasons and the increasing sample size does not change this fact. So, we can say that the estimators $\tilde{\rho}_k(\nu)$ and $\breve{\rho}_k(\nu)$ were more robust than the moment estimator $\hat{\rho}_k(\nu)$. This result is legible from Tables (3.3)-(3.5) and Figures (3.11) and (3.12).

In the two additive outliers case, the moment estimator $\hat{\rho}_k(\nu)$ has a greater bias and MSE at seasons one and two while this is not true for seasons three and four. For case three which represents the existence of four additive outliers, the moment estimator $\hat{\rho}_k(\nu)$ has the largest absolute bias and MSE for all seasons while the third estimator $\breve{\rho}_k(\nu)$ has the smallest absolute bias and MSE for all seasons (see Figures (3.15)-(3.18)). In general the second and third estimators are more robust than the moment estimator in the presence of outlier on the basis of bias and MSE, and the third estimator $\breve{\rho}_k(\nu)$ seems to be more robust than the second estimator $\tilde{\rho}_k(\nu)$ in almost all cases that are considered.

Also, it seems that the same results can be concluded for $PAR_{12}(1)$ model as can be seen from Tables (3.6)-(3.8) and Figures (3.13) - (3.14).

For Model 1, As time lag increases the absolute bias and MSE for $\hat{\rho}_k(\nu)$ decreases at season one (where the outlier exists) while this is not true for the other seasons. For $\tilde{\rho}_k(\nu)$ as time lag increases the absolute bias and MSE seems to be decreases in all seasons while this behavior not clear for $\breve{\rho}_k(\nu)$. For season one and for large lags the absolute bias and MSE seems to be very close for the three estimators, $\hat{\rho}_k(\nu)$, $\tilde{\rho}_k(\nu)$ and $\breve{\rho}_k(\nu)$.

Note that our results are true, namely for the selected models and cases.

CHAPTER 4

Robust Estimation of SACF for Higher PAR Models

4.1 Introduction

In this chapter, we are interested in robust estimation for the SACF for PAR models with higher orders. More specifically, we will try to generalize our work in chapter three to PAR models of orders larger than one.

Under periodic stationary, a general formula for PAR model with varying orders $p(\nu)$ is given by:

$$X_{t,\nu} - \mu_\nu = \sum_{i=1}^{p(\nu)} \phi_i(\nu)\left(X_{t,\nu-i} - \mu_{\nu-i}\right) + a_{t,\nu}, \tag{4.1}$$

where $\nu = 1,2\ldots.\omega$ denotes the season , t=1 ... stands for the years, μ_ν is the mean of season ν, $\{a_{t,\nu}\}$ is a periodic white noise process with zero mean and periodic variances σ_ν^2. If $p(\nu) = p$, $\forall$ ν then the model is the constant order PAR_ω (p) model.

4.2 SACF for PAR(2) Model

Let $\{X_{j\omega+\nu}\}$ follows a periodic stationary PAR(p) process, Hiple and Mcleod (1994) find a general formula for the SACVF for zero-mean PAR(p) models given by:

$$\gamma_k(\nu) = \phi_1(\nu)\gamma_{k-1}(\nu-1) + \phi_2(\nu)\gamma_{k-2}(\nu-2) + \ldots + \phi_p(\nu)\gamma_{k-p}(\nu-p) \quad k>0. \tag{4.2}$$

Note that if the model is non-periodic then,

$$\gamma_0(\nu) = \gamma_0(\nu-1) = \ldots = \gamma_0(\nu-p) = \gamma_0, \forall\, \nu = 1,2,..,\omega.$$

So that (4.2) reduces to,

$$\gamma_k = \phi_1\gamma_{k-1} + \phi_2\gamma_{k-2} + \ldots + \phi_p\gamma_{k-p} \quad k = 1,2,\ldots,p$$

Which is the ordinary Yule-walker equation for the AR(p) model (Cryer and Chan, 2008).

Theorem (4.1): If $\{X_{j\omega+\nu}\}$ follows a periodic stationary PAR$_\omega$(p) model, then the SACF at lag k; $\rho_k(\nu)$ is:

$$\rho_k(\nu)=\phi_1(\nu)\sqrt{\frac{\gamma_0(\nu-1)}{\gamma_0(\nu)}}\rho_{k-1}(\nu-1)+\phi_2(\nu)\sqrt{\frac{\gamma_0(\nu-2)}{\gamma_0(\nu)}}\rho_{k-2}(\nu-2)$$
$$+\ldots+\phi_p(\nu)\sqrt{\frac{\gamma_0(\nu-p)}{\gamma_0(\nu)}}\rho_{k-p}(\nu-p);\ \ k\geq 1 \tag{4.3}$$

where, $\rho_{-j}(\nu-p)=\rho_j(\nu-p+j);\ \ j=1,2,\ldots$.

Proof.

We know that,

$$\rho_k(\nu)=\frac{\gamma_k(\nu)}{\sqrt{\gamma_0(\nu)\gamma_0(\nu-k)}}$$

then dividing both sides of (4.2) by $\sqrt{\gamma_0(\nu)\gamma_0(\nu-k)}$, we will get;

$$\rho_k(\nu)=\phi_1(\nu)\frac{\gamma_{k-1}(\nu-1)}{\sqrt{\gamma_0(\nu)\gamma_0(\nu-k)}}+\phi_2(\nu)\frac{\gamma_{k-2}(\nu-2)}{\sqrt{\gamma_0(\nu)\gamma_0(\nu-k)}}+\ldots+\phi_p(\nu)\frac{\gamma_{k-p}(\nu-p)}{\sqrt{\gamma_0(\nu)\gamma_0(\nu-k)}}$$

So that,

$$\rho_k(\nu)=\phi_1(\nu)\sqrt{\frac{\gamma_0(\nu-1)}{\gamma_0(\nu)}}\rho_{k-1}(\nu-1)+\phi_2(\nu)\sqrt{\frac{\gamma_0(\nu-2)}{\gamma_0(\nu)}}\rho_{k-2}(\nu-2)$$
$$+\ldots+\phi_p(\nu)\sqrt{\frac{\gamma_0(\nu-p)}{\gamma_0(\nu)}}\rho_{k-p}(\nu-p);\ \ k\geq 1.$$

The above theorem is useful for the computation of the theoretical SACF of $PAR_\omega(p)$ models for given ϕ's. However the time lag of the SACF, $\rho_k(\nu)$, may some times be negative. To overcome this problem we firstly recall that the ordinary ACF, ρ_k is symmetric with respect to the time lag; i.e. $\rho_k = \rho_{-k}$ while the SACF, $\rho_k(\nu)$ is not, simply because the SACF is a function of time lag and season. It can be proved that if j > 0 then $\rho_{-j}(\nu) = \rho_j(\nu + j)$. For instance, assuming $\omega = 4$;

$$\begin{aligned}\rho_{-1}(3) &= Corr(X_{4k+3}, X_{4k+3-(-1)}) \\ &= Corr(X_{4k+3}, X_{4k+4}) \\ &= Corr(X_{4k+4}, X_{4k+3}) \\ &= \rho_1(4).\end{aligned}$$

The following corollary is a straitforward result from Theorem (4.1) above for the $PAR_\omega(2)$ model.

Corollary (4.1): If $\{X_{j\omega+\nu}\}$ follows a periodic stationary $PAR_\omega(2)$ model, then the SACF at lag k; $\rho_k(\nu)$ is:

$$\rho_k(\nu) = \phi_1(\nu)\sqrt{\frac{\gamma_0(\nu-1)}{\gamma_0(\nu)}}\rho_{k-1}(\nu-1) + \phi_2(\nu)\sqrt{\frac{\gamma_0(\nu-2)}{\gamma_0(\nu)}}\rho_{k-2}(\nu-2). \qquad (4.4)$$

Now to clarify the usage of Theorem (4.1) and Corollary (4.1) for the computation of the theoretical SACF of $PAR_\omega(p)$ models, we consider the zero-mean $PAR_4(2)$ model defined as:

$$\left.\begin{aligned} X_{j4+1} &= \phi_1(1)X_{(j-1)4+4} + \phi_2(1)X_{(j-1)4+3} + a_{j4+1} \\ X_{j4+2} &= \phi_1(2)X_{j4+1} + \phi_2(2)X_{(j-1)4+4} + a_{j4+2} \\ X_{j4+3} &= \phi_1(3)X_{j4+2} + \phi_2(3)X_{j4+1} + a_{j4+3} \\ X_{j4+4} &= \phi_1(4)X_{j4+3} + \phi_2(4)X_{j4+2} + a_{j4+4} \end{aligned}\right\} . \qquad (4.5)$$

We have seen previously that the $\text{PAR}_\omega(1)$ model is periodic stationary if $\left|\prod_{\nu=1}^{\omega}\phi_1(\nu)\right|<1$. For higher PAR models checking periodic stationarity is a bit more difficult job. Ula and Smadi (1997) proposed a methodology for this objective by utilizing the lumped processes approach so that the periodic stationarity conditions became an eigen value problem.

Following Ula and Smadi (1997), the $\text{PAR}_4(2)$ is periodic stationary if all the eigen-values of the matrix $R = L^{-1}U_1$ be less than one in modulus, where,

$$\text{L} = \begin{cases} 1 & \text{i} = \text{j} \\ 0 & \text{i} < \text{j} \\ -\phi_{\text{i-j}}(i) & \text{o.w} \end{cases} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -\phi_1(2) & 1 & 0 & 0 \\ -\phi_2(3) & -\phi_1(3) & 1 & 0 \\ 0 & -\phi_2(4) & -\phi_1(4) & 1 \end{bmatrix}$$

and

$$\text{U}_1 = \phi_{\omega+i-j}(i) = \begin{bmatrix} 0 & 0 & \phi_2(1) & \phi_1(1) \\ 0 & 0 & 0 & \phi_2(2) \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} .$$

Therefore, before the computing of $\rho_k(v)$ we should check the periodic stationary of the model. By assuming the stationarity and to find $\rho_k(v)$ by using (4.4), we need the values of $\gamma_0(1), \gamma_0(2), \gamma_0(3), \gamma_0(4)$, and $\gamma_1(1), \gamma_1(2), \gamma_1(3), \gamma_1(4)$. The following Lemma helps us to find these values.

Lemma (4.1): Let $\{X_{j\omega+\nu}\}$ follows a periodic stationary $PAR_4(2)$ model, then from (4.5) we have the following equations;

$$\gamma_0(1) = (\phi_1(1))^2\gamma_0(4) + (\phi_2(1))^2\gamma_0(3) + 2\phi_1(1)\phi_2(1)\gamma_1(4) + \sigma_1^2.$$

$$\gamma_1(1) = \phi_1(1)\gamma_0(4) + \phi_2(1)\gamma_1(4).$$

$$\gamma_0(2) = (\phi_1(2))^2\gamma_0(1) + (\phi_2(2))^2\gamma_0(4) + 2\phi_1(2)\phi_2(2)\gamma_1(1) + \sigma_2^2.$$

$$\gamma_1(2) = \phi_1(2)\gamma_0(1) + \phi_2(2)\gamma_1(1).$$

$$\gamma_0(3) = (\phi_1(3))^2\gamma_0(2) + (\phi_2(3))^2\gamma_0(1) + 2\phi_1(3)\phi_2(3)\gamma_1(2) + \sigma_3^2.$$

$$\gamma_1(3) = \phi_1(3)\gamma_0(2) + \phi_2(3)\gamma_1(2).$$

$$\gamma_0(4) = (\phi_1(4))^2\gamma_0(3) + (\phi_2(4))^2\gamma_0(2) + 2\phi_1(4)\phi_2(4)\gamma_1(3) + \sigma_4^2.$$

$$\gamma_1(4) = \phi_1(4)\gamma_0(3) + \phi_2(4)\gamma_1(3).$$

Proof.

We can find $\gamma_0(\nu); \nu = 1,2,3,4$ by taking the variance for each equation of (4.5), and the values of $\gamma_1(\nu); \nu = 1,2,3,4$ by multiplying each equation of (4.5) by the preceding value, i.e. $X_{t,\nu-1}$, then taking the expectation.

From the above system of equations we can find the unknowns, $\gamma_0(1), \gamma_0(2), \gamma_0(3), \gamma_0(4)$, $\gamma_1(1), \gamma_1(2), \gamma_1(3)$ and $\gamma_1(4)$ for given ϕ's and σ_ν^2's.

Example 4.1:

Consider the zero-mean $PAR_4(2)$ model with ϕ_1's = -0.1, 0.8, 0.95, 1.1, ϕ_2's = 0.8, 0.4, -0.7, 0.3 and σ_ν^2's :1, 64, 4, 9. To check the stationarity for this model we need the values of $R = L^{-1}U_1$

$$L=\begin{bmatrix}1 & 0 & 0 & 0\\ -0.8 & 1 & 0 & 0\\ 0.7 & -0.95 & 1 & 0\\ 0 & -0.3 & -1.1 & 1\end{bmatrix}, \quad U_1=\begin{bmatrix}0 & 0 & 0.8 & -0.1\\ 0 & 0 & 0 & 0.4\\ 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\end{bmatrix}$$

and,

$$R=L^{-1}U_1=\begin{bmatrix}0 & 0 & 0.8 & -0.1\\ 0 & 0 & 0.64 & 0.32\\ 0 & 0 & 0.048 & 0.374\\ 0 & 0 & 0.2448 & 0.5074\end{bmatrix}.$$

Therefore, the eigen-values of R are: 0.6575912, -0.1021912, 0.0, 0.0. Since all eigen-values are less than one in modulus, the $PAR_4(2)$ model is periodic stationary.

By using Lemma (4.1), the vales of $\gamma_0(\nu);\nu=1,2,3,4$ and $\gamma_1(\nu);\nu=1,2,3,4$ are summarized in Table (4.1). Also, by using Corollary (4.1), the values of the theoretical SACF are computed and given in Table (4.2). They represented in Figure (4.1). It can be seen that the theoretical values of SACF for $PAR_4(2)$ dies out as the lag k increases for the four seasons.

Table (4.1): The values of $\gamma_0(\nu)$ and $\gamma_1(\nu)$ for the $PAR_4(2)$ model

$\gamma_0(1)$	$\gamma_0(2)$	$\gamma_0(3)$	$\gamma_0(4)$	$\gamma_1(1)$	$\gamma_1(2)$	$\gamma_1(3)$	$\gamma_1(4)$
43.44	190.27	99.85	232.50	95.73	73.04	129.63	148.73

Table (4.2): The theoretical values of SACF, $\rho_k(v)$ for the $PAR_4(2)$ model

	Season			
Time lag	1	2	3	4
0	1.000	1.000	1.000	1.000
1	0.9526	0.8035	0.9404	0.9761
2	0.9871	0.8063	0.5919	0.9493
3	0.9211	0.8089	0.6175	0.6448
4	0.5688	0.7718	0.6051	0.664
5	0.5954	0.5025	0.5869	0.6557
6	0.5822	0.5212	0.3964	0.6326
7	0.5656	0.5125	0.4086	0.4221
8	0.3831	0.4959	0.4032	0.436
9	0.3947	0.3331	0.3892	0.4298
10	0.3897	0.3437	0.2599	0.4151

And the graph of the four seasons for the ten lag is:

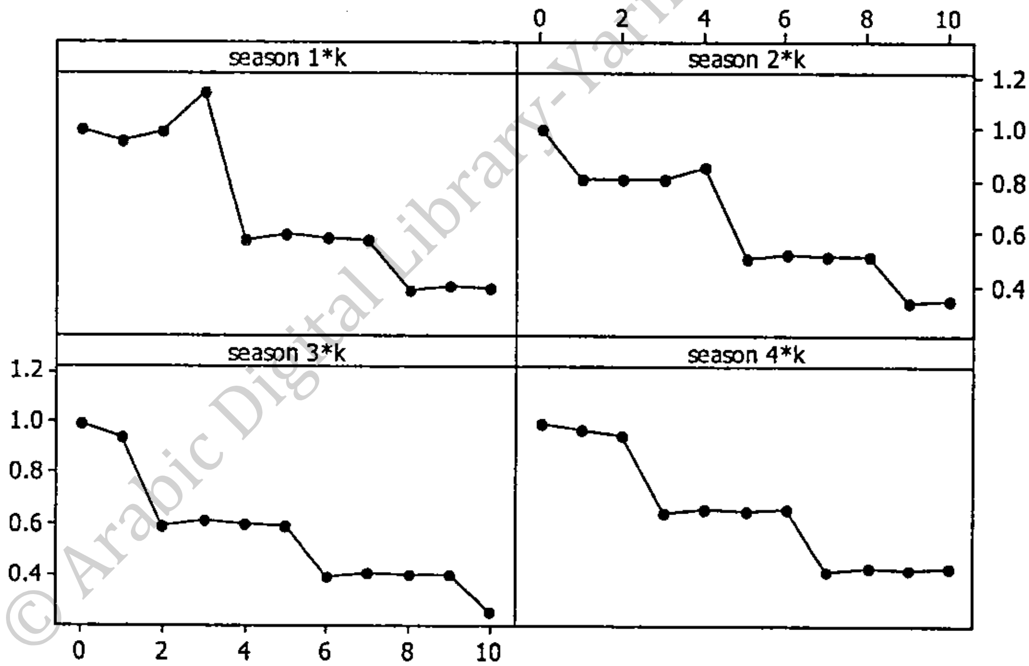

Figure (4.1): The theoretical values of SACF for $PAR_4(2)$ model

4.3 Robust Estimation of $\rho_k(\nu)$ in PAR(2) Model

In this section we will generalize the estimators of SACF which are given by formulas (3.4) and (3.6) for $\text{PAR}_4(2)$ model. The estimator in (3.5) will not be considered since it is not easy job to generalize it for the $\text{PAR}_4(2)$ model.

Assuming that $\{X_{t,\nu}\}$ follows $\text{PAR}_4(2)$ model as defined in (4.5) and define $\{Z_{t,\nu}\}$ to be the same as $\{X_{t,\nu}\}$ but contaminated with an additive outlier Δ at year t_0 and season ν_0, i.e,

$$Z_{t,\nu} = \begin{cases} X_{t,\nu}, & (t,\nu) \neq (t_0,\nu_0) \\ X_{t,\nu} + \Delta, & (t,\nu) = (t_0,\nu_0) \end{cases} .$$

Accordingly, we aim to estimate the SACF of $\{Z_{t,\nu}\}$; i.e. $\rho_k(\nu)$; $k \geq 1$. The proposed estimators for $\rho_k(\nu)$ are:

$$\hat{\rho}_k(\nu) = \frac{\sum_{t=1}^{n} (Z_{t,\nu} - \bar{Z}_\nu)(Z_{t,\nu-k} - \bar{Z}_{\nu-k})}{\sqrt{\sum_{t=1}^{n} (Z_{t,\nu} - \bar{Z}_\nu)^2 \sum_{t=1}^{n} (Z_{t,\nu-k} - \bar{Z}_{\nu-k})^2}} \qquad (4.6)$$

$$\tilde{\rho}_k(\nu) = \frac{\underset{t}{Med}\{Z^*_{t,\nu} Z^*_{t,\nu-k}\}}{\sqrt{\underset{t}{Med}\{Z^{*2}_{t,\nu}\} \underset{t}{Med}\{Z^{*2}_{t,\nu-k}\}}} \qquad (4.7)$$

where $Z^*_{t,\nu} = Z_{t,\nu} - Med\{Z_{t,\nu}\}, t = 1,\ldots,n$ is the seasonally median subtracted from time series and $\underset{t}{Med}\{Z_{t,\nu}\}$ is the median of the data in season $\nu; \nu = 1,\ldots.\omega$.

4.4 Varying Order PAR Model

Consider the $PAR_{\omega}(p(\nu))$ model of varying orders in (4.1). In this section we will study the estimation of the SACF of such models. Here we consider the zero-mean $PAR_4(2,1,0,2)$ model defined as:

$$\left.\begin{aligned} X_{j4+1} &= \phi_1(1)X_{(j-1)4+4} + \phi_2(1)X_{(j-1)4+3} + a_{j4+1} \\ X_{j4+2} &= \phi_1(2)X_{j4+1} + a_{j4+2} \\ X_{j4+3} &= a_{j4+3} \\ X_{j4+4} &= \phi_1(4)X_{j4+3} + \phi_2(4)X_{j4+2} + a_{j4+4} \end{aligned}\right\} \quad (4.8)$$

To check if this model is periodic stationary a similar approach to the $PAR_4(2)$ model discussed previously can be applied.

We will deal with this model as a $PAR_4(2)$ model but replacing the missing parameters with zeros, i.e. $\phi_2(2)$, $\phi_1(3)$ and $\phi_2(3)$ are all set to zero. Thus we will use Lemma (4.1) to find $\gamma_0(\nu); \nu = 1,2,3,4$ and $\gamma_1(\nu); \nu = 1,2,3,4$ and Theorem (4.1) along Corollary (4.1) to compute $\rho_k(\nu)$ for $PAR_4(2,1,0,2)$ model.

Example 4.2:

Consider the zero-mean $PAR_4(2,1,0,2)$ model with ϕ_1's = -0.1, 0.8, 0.0, 1.1, ϕ_2's = 0.8, 0.0, 0.0, 0.3 and σ_ν^2's :1, 64, 4, 9.

To check the stationarity for this model we need the values of $R = L^{-1}U_1$

$$L = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -0.8 & 1 & 0 & 0 \\ 0.0 & 0.0 & 1 & 0 \\ 0 & -0.3 & -1.1 & 1 \end{bmatrix} \quad , \quad U_1 = \begin{bmatrix} 0 & 0 & 0.8 & -0.1 \\ 0 & 0 & 0 & 0.0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

and,

$$R = L^{-1}U_1 = \begin{bmatrix} 0 & 0 & 0.8 & -0.1 \\ 0 & 0 & 0.64 & -0.08 \\ 0 & 0 & 0.0 & 0.0 \\ 0 & 0 & 0.192 & -0.024 \end{bmatrix}.$$

Thus the eigen-values of R are: -0.024, 0.0, 0.0, 0.0. Since all eigen-values are less than one in modulus, then this $PAR_4(2,1,0,2)$ model is periodic stationary.

Using Lemma (4.1), the values of $\gamma_0(\nu); \nu = 1,2,3,4$ and $\gamma_1(\nu); \nu = 1,2,3,4$ are computed and summarized in Table (4.3). Besides, using Corollary (4.1), the values of the theoretical SACF are computed and summarized in Table (4.4). A sketch of it is given in Figure (4.2).

Table (4.3): The values of $\gamma_0(\nu)$ and $\gamma_1(\nu)$ for $PAR_4(2,1,0,2)$ model

$\gamma_0(1)$	$\gamma_0(2)$	$\gamma_0(3)$	$\gamma_0(4)$	$\gamma_1(1)$	$\gamma_1(2)$	$\gamma_1(3)$	$\gamma_1(4)$
3.054	65.954	4.000	19.776	1.542	2.443	0.000	4.400

Table (4.4): The theoretical values of SACF, $\rho_k(\nu)$ for $PAR_4(2,1,0,2)$ model

	Season			
Time lag	**1**	**2**	**3**	**4**
0	1.000	1.000	1.000	1.000
1	0.1985	0.1721	0	0.4947
2	0.7897	0.0342	0	0.5479
3	-0.1394	0.1359	0	0.0943
4	-0.0240	-0.0240	0	0.0187
5	-0.0048	-0.0041	0	0.0745
6	-0.0190	-0.0008	0	-0.0131
7	0.0033	-0.0033	0	-0.0023
8	0.0006	0.0006	0	-0.0004
9	0.0001	0.0001	0	-0.0018
10	0.0005	0.0000	0	0.0003

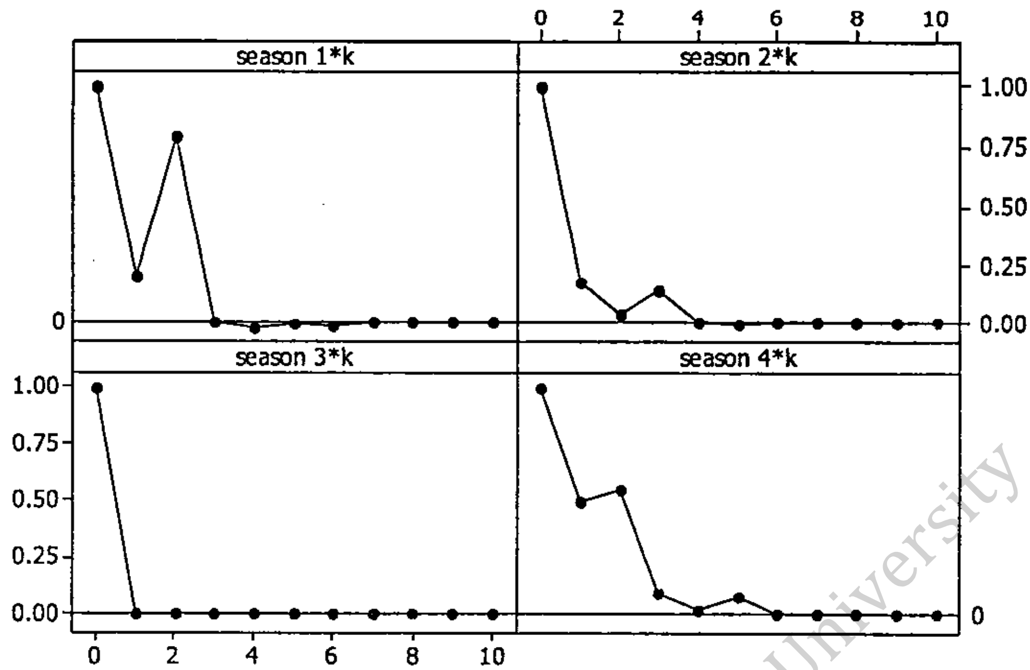

Figure (4.2): The theoretical values of SACF for $PAR_4(2,1,0,2)$ model

In Figure (4.2), the theoretical SACF for $PAR_4(2,1,0,2)$ model decreases and approach to zero as time lag increases in the first, second and fourth seasons. While in season three there is a cut-off after lag zero which is the behavior of the white noise.

As far as the estimation of $\rho_k(\nu)$ for $PAR_4(2,1,0,2)$ model is done, we will use $\hat{\rho}_k(\nu)$ and $\breve{\rho}_k(\nu)$ as defined by (4.6) and (4.7).

4.5 Methodology and Simulation Results

In this section we used the Monte-Carlo technique to study the performance of $\hat{\rho}_k(\nu)$ and $\breve{\rho}_k(\nu)$ of $\rho_k(\nu)$ for $PAR_4(2)$ and $PAR_4(2,1,0,2)$ models through absolute mean bias and MSE. Regarding the additive outliers we will investigate the following cases:

(1) A single additive outlier at t=13 (season one) and $\Delta = 100$.

(2) Five additive outliers with random magnitudes (assumed U(50,400)) and random positions (discrete $U(1,\omega n)$) where n is the number of years.

As far as the simulation-work is concerned; 1000 realizations each of length n years (30, 50,100) will be simulated from our models assuming that the white noise process is normal with mean zero and periodic variances σ_ν^2. In each simulated realization the estimate of $\rho_1(\nu)$ is computed, say $\hat{\rho}_1(\nu)$. Then, based on the 1000 realization, the absolute bias and MSE are computed (which are previously defined in chapter 3).

In order to investigate the behavior of the two estimators $\hat{\rho}_1(\nu)$ and $\breve{\rho}_k(\nu)$ we will carry out the simulation based on the following models:

(1) $PAR_4(2)$ with ϕ_1's : -0.1, 0.8, 0.95, 1.1. and ϕ_2's : 0.8, 0.4, -0.7, 0.3.

and σ_ν^2's: 1, 64, 4, 9.

(2) $PAR_4(2,1,0,2)$ with ϕ_1's : -0.1, 0.8, 0, 1.1. and ϕ_2's : 0.8, 0, 0, 0.3.

and σ_ν^2's: 1, 64, 4, 9.

In what follows the model in part (1) will be referred as Model 1 and the model in part (2) is referred as Model 2.

All of the PAR models above are chosen to be periodic stationary (see Examples 4.1 and 4.2).

Due to the large amount of results, we have summarized the most important results in Tables (4.5)-(4.10) and Figures (4.3)-(4.14). The results corresponding to Model 1 are summarized in Tables (4.5)-(4.7) and Figures (4.3)-(4.8), whereas those of Model 2 are presented in Tables (4.8)-(4.10) and Figures (4.9)-(4.14).

Tables (4.5)-(4.7) presents the absolute bias and MSE (in brackets) of the two estimators of the SACF of Model 1 with additive outlier at season one for n=30, 50 and 100. Figures (4.3)-(4.6) present the absolute bias and MSE for each estimator of Model 1 separately. Figures (4.7)-(4.8) present the absolute bias and MSE of the two estimators of the SACF of Model 1 with additive outlier at season one for n=30.

Tables (4.8)-(4.10) present the absolute bias and MSE (in brackets) of the two estimators of the SACF of Model 2 with five additive outliers for n=30, 50 and 100. Figures (4.9)-(4.12) present the absolute bias and MSE for each estimator of Model 2 separately. While Figures (4.13) and (4.14) present the absolute bias and MSE of the two estimators of the SACF of Model 2 with five additive outliers and n=30.

Table (4.5): The absolute bias and MSE (in brackets) of the two estimators of the SACF of the $PAR_4(2)$ model (Model 1), n = 30 with additive outlier at season one

	Season							
	1		2		3		4	
Lag	$\hat{\rho}_k$	$\breve{\rho}_k$	$\hat{\rho}_k$	$\breve{\rho}_k$	$\hat{\rho}_k$	$\breve{\rho}_k$	$\hat{\rho}_k$	$\breve{\rho}_k$
1	0.6682 (0.4776)	0.0189 (0.0781)	0.5627 (0.3488)	0.1004 (0.1306)	0.0176 (0.0013)	0.0095 (0.0665)	0.0071 (0.0003)	0.0665 (0.0585)
2	0.6884 (0.5047)	0.0297 (0.0634)	0.0943 (0.0205)	0.1419 (0.1175)	0.4294 (0.2200)	0.2372 (0.1507)	0.0157 (0.0012)	0.0187 (0.0717)
3	0.6487 (0.4522)	0.0580 (0.0782)	0.0910 (0.0192)	0.1313 (0.1158)	0.1307 (0.0463)	0.2725 (0.1473)	0.4628 (0.2500)	0.2273 (0.1491)
4	0.5558 (0.3181)	0.2579 (0.1693)	0.1084 (0.0262)	0.1773 (0.1334)	0.1307 (0.0476)	0.2697 (0.1570)	0.1252 (0.0404)	0.2626 (0.1551)
5	0.4525 (0.2411)	0.2945 (0.1567)	0.3780 (0.1787)	0.2570 (0.1395)	0.1410 (0.0517)	0.2900 (0.1517)	0.1216 (0.0401)	0.2612 (0.1519)
6	0.4446 (0.2346)	0.2897 (0.1595)	0.1683 (0.0657)	0.2767 (0.1463)	0.3085 (0.1317)	0.2532 (0.1154)	0.1342 (0.0459)	0.2797 (0.1570)
7	0.4328 (0.2255)	0.3022 (0.1585)	0.1654 (0.0639)	0.2722 (0.1412)	0.1614 (0.0660)	0.2765 (0.1187)	0.3222 (0.1403)	0.2638 (0.1218)
8	0.3907 (0.1608)	0.2506 (0.1167)	0.1812 (0.0753)	0.2808 (0.1509)	0.1571 (0.0634)	0.2659 (0.1194)	0.1673 (0.0683)	0.2807 (0.1322)
9	0.3143 (0.1393)	0.2689 (0.1151)	0.2731 (0.1117)	0.2351 (0.0991)	0.1712 (0.0722)	0.2706 (0.1149)	0.1610 (0.0651)	0.2772 (0.1238)
10	0.3096 (0.1363)	0.2654 (0.1125)	0.1867 (0.0785)	0.2463 (0.1034)	0.2184 (0.0842)	0.2074 (0.0717)	0.1762 (0.0751)	0.2800 (0.1239)

Table (4.6): The absolute bias and MSE (in brackets) of the two estimators of the SACF of the $PAR_4(2)$ model (Model 1), n = 50 with additive outlier at season one

	Season							
	1		2		3		4	
Lag	$\hat{\rho}_k$	$\breve{\rho}_k$	$\hat{\rho}_k$	$\breve{\rho}_k$	$\hat{\rho}_k$	$\breve{\rho}_k$	$\hat{\rho}_k$	$\breve{\rho}_k$
1	0.5736 (0.3447)	0.0179 (0.0289)	0.4856 (0.2534)	0.1082 (0.0634)	0.0107 (0.0006)	0.0070 (0.0297)	0.0043 (0.0001)	0.0267 (0.0200)
2	0.5937 (0.3678)	0.0111 (0.0192)	0.0582 (0.0096)	0.1148 (0.0629)	0.3705 (0.1581)	0.2138 (0.1019)	0.0100 (0.0006)	0.0007 (0.0287)
3	0.5576 (0.3267)	0.0517 (0.0361)	0.0548 (0.0087)	0.1206 (0.0630)	0.0813 (0.0234)	0.2237 (0.1053)	0.4008 (0.1806)	0.2025 (0.0981)
4	0.5005 (0.2586)	0.2254 (0.1098)	0.0666 (0.0122)	0.1462 (0.0797)	0.0797 (0.0237)	0.2228 (0.1086)	0.0787 (0.0203)	0.2066 (0.1046)
5	0.3825 (0.1662)	0.2367 (0.1113)	0.3315 (0.1330)	0.2147 (0.0981)	0.0873 (0.0262)	0.2356 (0.1074)	0.0751 (0.0199)	0.2099 (0.0994)
6	0.3741 (0.1605)	0.2386 (0.1118)	0.1055 (0.0338)	0.2290 (0.1052)	0.2682 (0.0957)	0.2102 (0.0828)	0.0835 (0.0230)	0.2267 (0.1066)
7	0.3648 (0.1533)	0.2423 (0.1103)	0.1033 (0.0329)	0.2303 (0.1031)	0.1009 (0.0356)	0.2295 (0.0898)	0.2831 (0.1041)	0.2159 (0.0868)
8	0.3485 (0.1292)	0.2083 (0.0810)	0.1132 (0.0387)	0.2303 (0.1066)	0.0985 (0.0348)	0.2259 (0.0889)	0.1038 (0.0360)	0.2320 (0.0986)
9	0.2662 (0.0938)	0.2295 (0.0864)	0.2400 (0.0826)	0.1950 (0.0700)	0.1065 (0.0391)	0.2275 (0.0871)	0.1002 (0.0349)	0.2311 (0.0927)
10	0.2610 (0.0911)	0.2287 (0.0847)	0.1184 (0.0446)	0.2100 (0.0759)	0.1943 (0.0621)	0.1736 (0.0513)	0.1083 (0.0398)	0.2315 (0.0924)

Table (4.7): The absolute bias and MSE (in brackets) of the two estimators of the SACF of the $PAR_4(2)$ model (Model 1), n = 100 with additive outlier at season one

	Season							
	1		2		3		4	
Lag	$\hat{\rho}_k$	$\breve{\rho}_k$	$\hat{\rho}_k$	$\breve{\rho}_k$	$\hat{\rho}_k$	$\breve{\rho}_k$	$\hat{\rho}_k$	$\breve{\rho}_k$
1	0.4475 (0.2062)	0.0284 (0.0119)	0.3812 (0.1528)	0.1029 (0.0305)	0.0050 (0.0002)	0.0231 (0.0114)	0.0019 (0.0000)	0.0007 (0.0071)
2	0.4615 (0.2185)	0.0042 (0.0067)	0.0274 (0.0033)	0.1036 (0.0317)	0.2917 (0.0953)	0.1769 (0.0580)	0.0047 (0.0002)	0.0147 (0.0116)
3	0.4323 (0.1928)	0.0449 (0.0159)	0.0265 (0.0031)	0.1034 (0.0308)	0.0383 (0.0089)	0.1806 (0.0588)	0.3146 (0.1080)	0.1658 (0.0525)
4	0.4213 (0.1840)	0.1859 (0.0628)	0.0322 (0.0045)	0.1239 (0.0404)	0.0386 (0.0095)	0.1794 (0.0603)	0.0372 (0.0077)	0.1668 (0.0560)
5	0.2935 (0.0965)	0.1911 (0.0623)	0.2576 (0.0776)	0.1848 (0.0574)	0.0424 (0.0106)	0.1853 (0.0602)	0.0365 (0.0079)	0.1707 (0.0551)
6	0.2883 (0.0935)	0.1896 (0.0628)	0.0534 (0.0142)	0.1901 (0.0620)	0.2074 (0.0548)	0.1850 (0.0547)	0.0407 (0.0093)	0.1775 (0.0562)
7	0.2823 (0.0906)	0.1957 (0.0636)	0.0526 (0.0140)	0.1900 (0.0600)	0.0522 (0.0163)	0.1912 (0.0579)	0.2194 (0.0600)	0.1887 (0.0567)
8	0.2918 (0.0913)	0.1861 (0.0551)	0.0574 (0.0167)	0.1910 (0.0654)	0.0511 (0.0158)	0.1922 (0.0585)	0.0538 (0.0162)	0.1990 (0.0625)
9	0.2064 (0.0547)	0.1942 (0.0578)	0.1788 (0.0442)	0.1721 (0.0485)	0.0550 (0.0182)	0.1885 (0.0570)	0.0523 (0.0158)	0.1996 (0.0613)
10	0.2040 (0.0535)	0.1943 (0.0580)	0.0593 (0.0208)	0.1805 (0.0524)	0.1411 (0.0321)	0.1520 (0.0370)	0.0567 (0.0184)	0.1973 (0.0606)

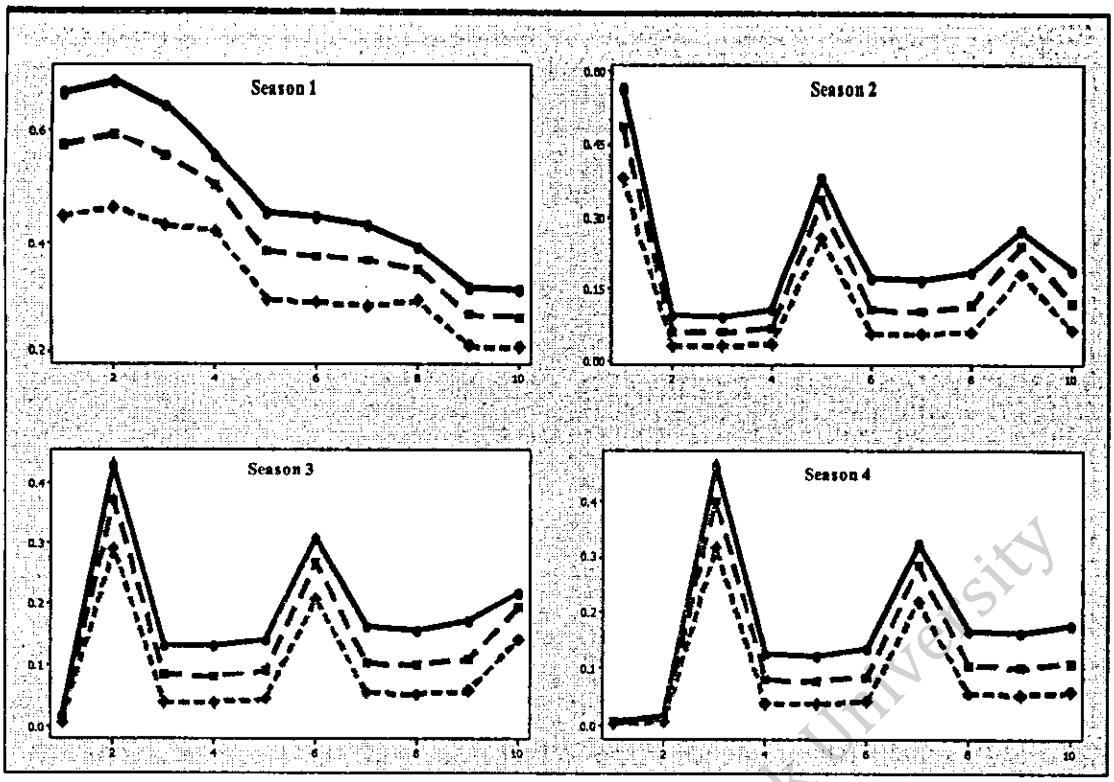

Figure (4.3): The absolute bias for $\hat{\rho}_k(\nu)$ for $PAR_4(2)$ model (Model 1) with a single additive outlier at season one (—: n=30, – – : n=50, ---: n=100)

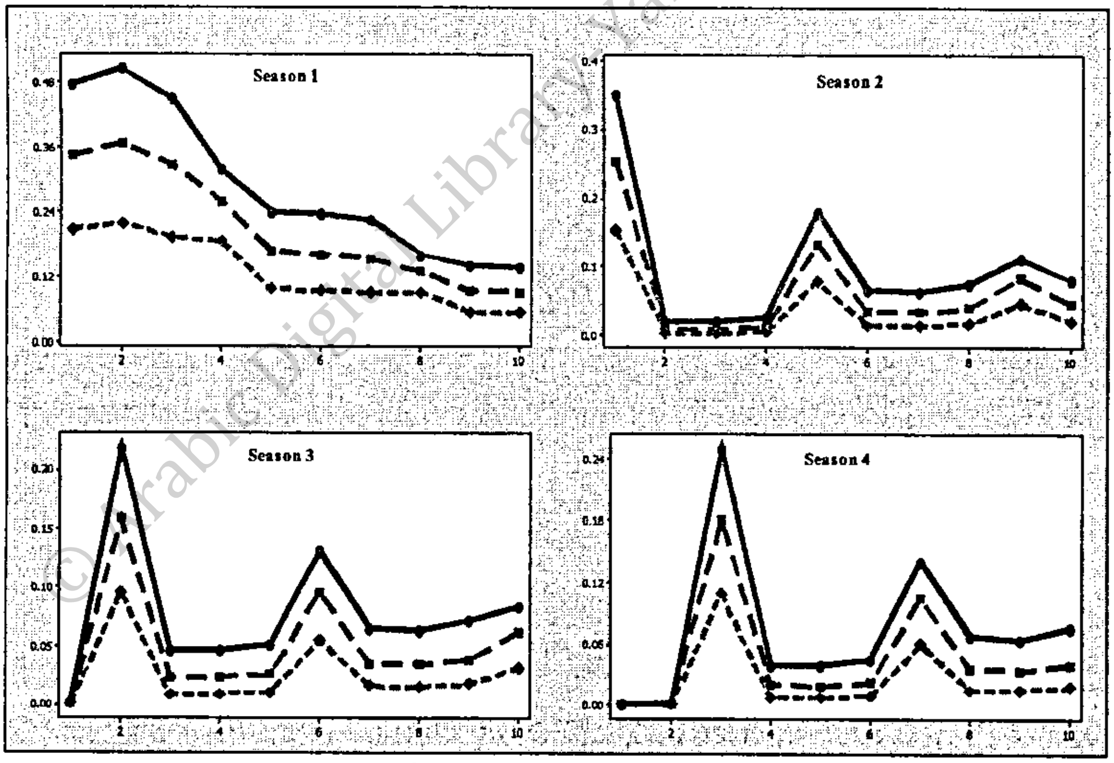

Figure (4.4): The MSE for $\hat{\rho}_k(\nu)$ for $PAR_4(2)$ model (Model 1) with a single additive outlier at season one (—: n=30, – – : n=50, ---: n=100)

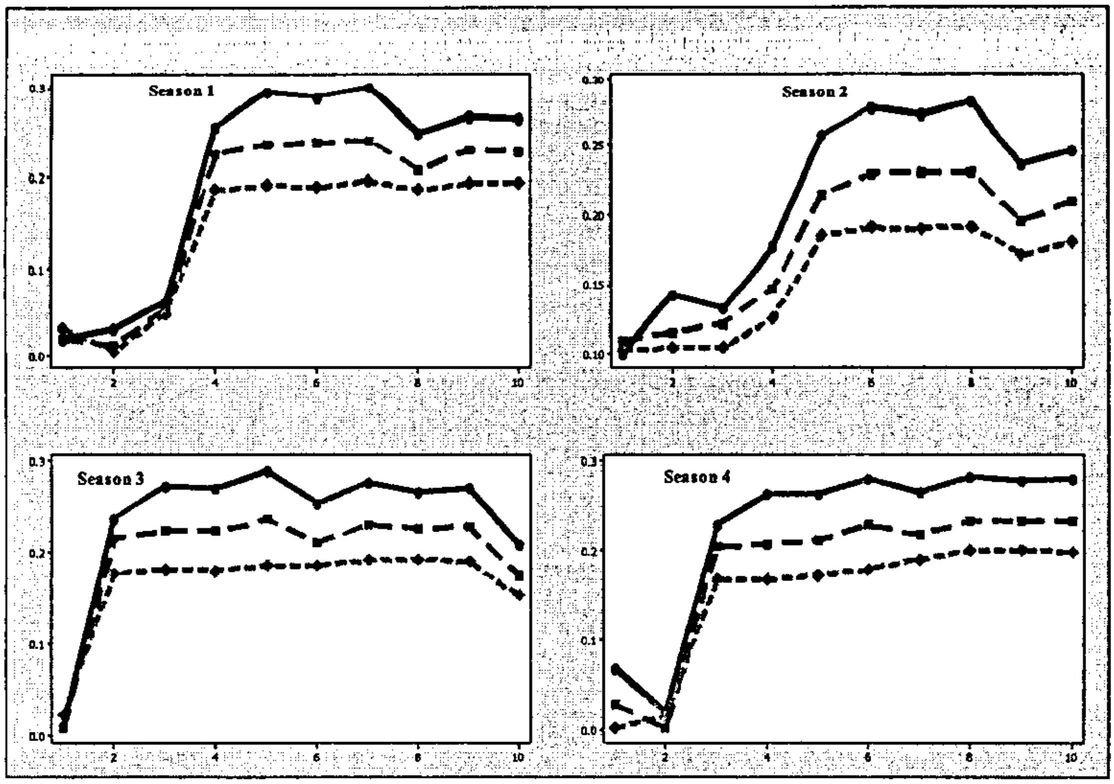

Figure (4.5): The absolute bias for $\breve{\rho}_k(\nu)$ for $PAR_4(2)$ model (Model 1) with a single additive outlier at season one (—: n=30, – – : n=50, ----: n=100)

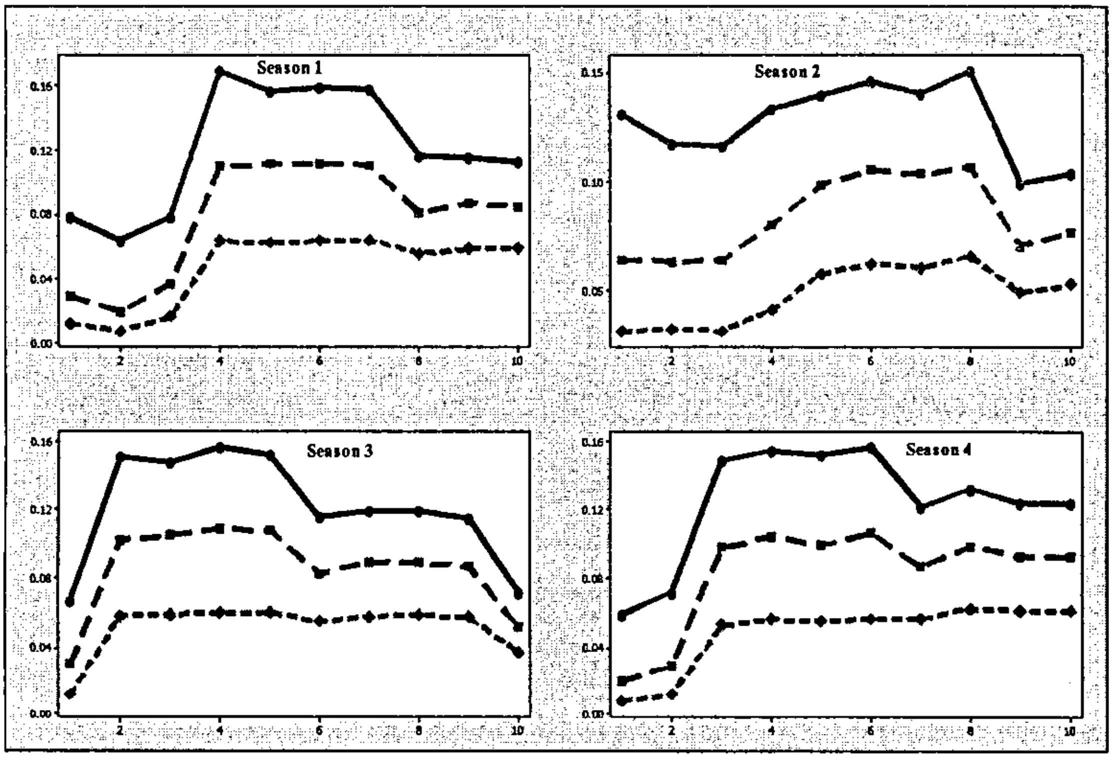

Figure (4.6): The MSE for $\breve{\rho}_k(\nu)$ for $PAR_4(2)$ model (Model 1) with a single additive outlier at season one (—: n=30, – – : n=50, ----: n=100)

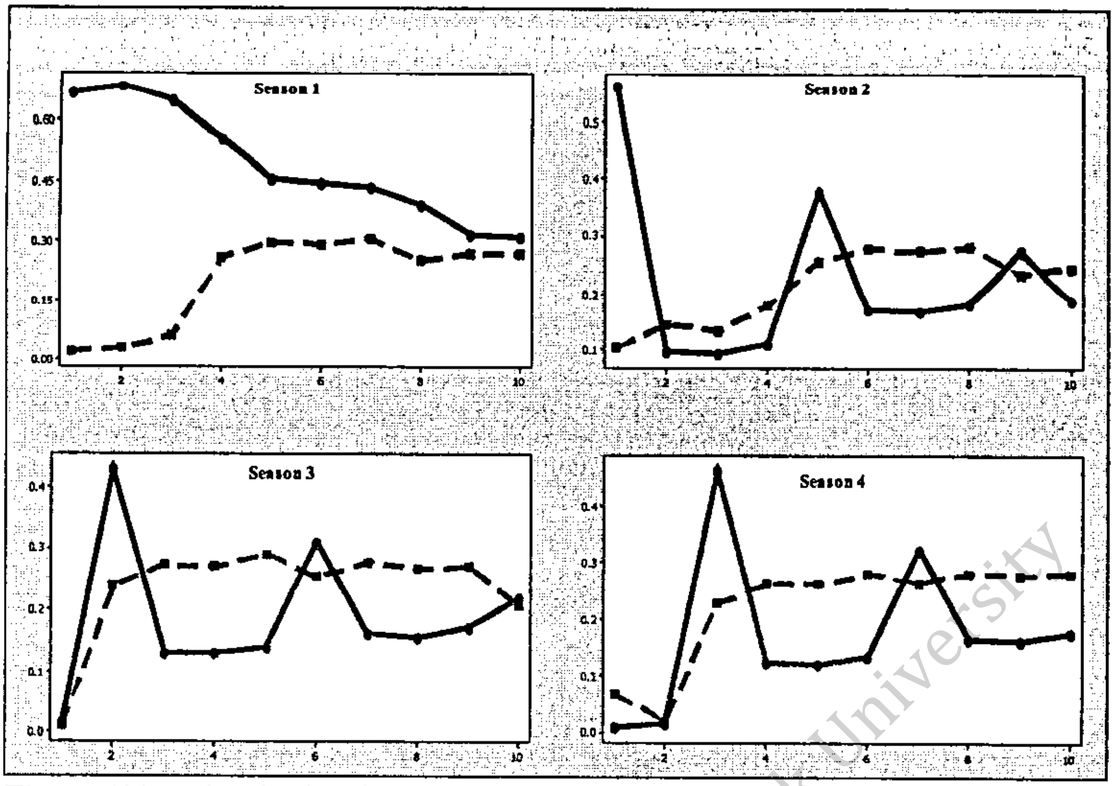

Figure (4.7): The absolute bias for the two estimators for $PAR_4(2)$ model (Model 1) with a single additive outlier at season one and n=30 (——: $\hat{\rho}_k(\nu)$, – – : $\breve{\rho}_k(\nu)$)

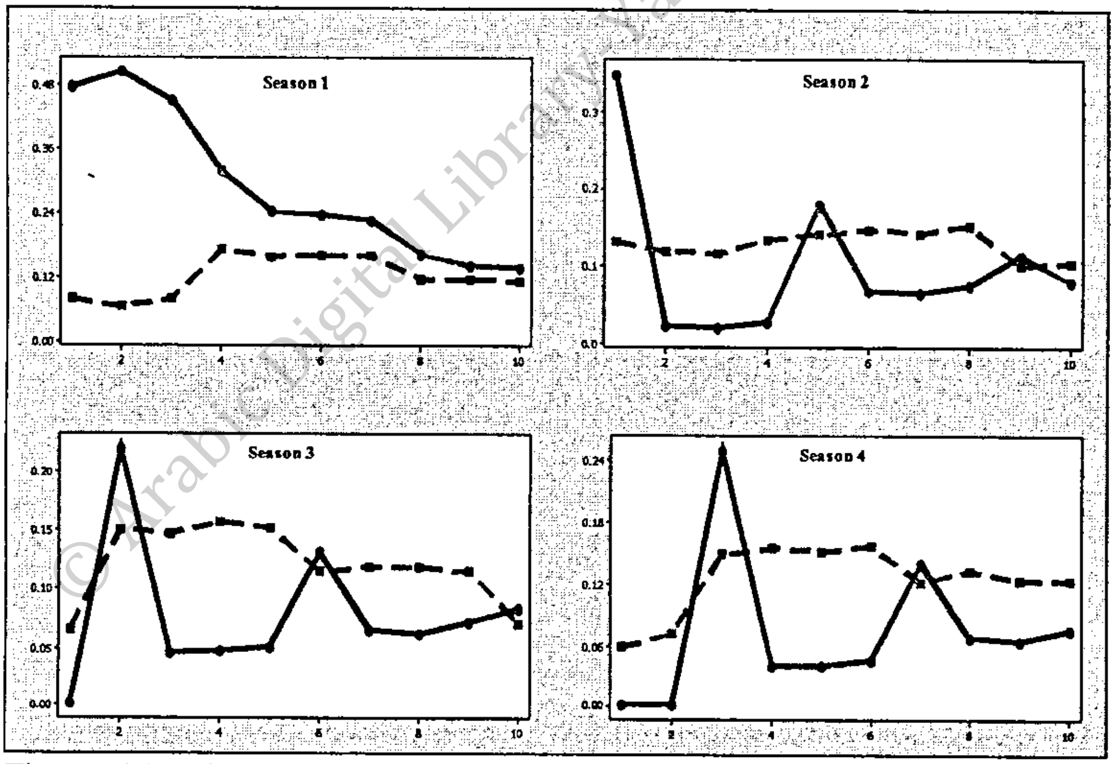

Figure (4.8): The MSE for the two estimators for $PAR_4(2)$ model (Model 1) with a single additive outlier at season one and n=30 (——: $\hat{\rho}_k(\nu)$, – – : $\breve{\rho}_k(\nu)$)

Table (4.8): The absolute bias and MSE (in brackets) of the two estimators of the SACF of the $PAR_4(2,1,0,2)$ model (Model 2), n = 30 with five additive outliers

	Season							
	1		2		3		4	
Lag	$\hat{\rho}_k$	$\breve{\rho}_k$	$\hat{\rho}_k$	$\breve{\rho}_k$	$\hat{\rho}_k$	$\breve{\rho}_k$	$\hat{\rho}_k$	$\breve{\rho}_k$
1	0.1953 (0.0727)	0.0936 (0.0509)	0.1620 (0.0644)	0.0596 (0.0587)	0.0092 (0.0276)	0.0008 (0.0354)	0.4704 (0.2594)	0.1767 (0.1087)
2	0.7564 (0.6196)	0.1483 (0.1301)	0.0305 (0.0351)	0.0330 (0.0405)	0.0063 (0.0418)	0.0203 (0.0356)	0.5037 (0.2932)	0.1882 (0.1179)
3	0.1250 (0.0472)	0.0553 (0.0482)	0.1258 (0.0471)	0.0670 (0.0409)	0.0020 (0.0344)	0.0085 (0.0244)	0.0828 (0.0416)	0.0495 (0.0414)
4	0.0140 (0.0120)	0.0354 (0.0491)	0.0166 (0.0154)	0.0236 (0.0443)	0.0342 (0.0133)	0.0337 (0.0344)	0.0436 (0.0148)	0.0286 (0.0293)
5	0.0070 (0.0337)	0.0048 (0.0248)	0.0129 (0.0265)	0.0062 (0.0296)	0.0010 (0.0317)	0.0118 (0.0290)	0.0675 (0.0352)	0.0539 (0.0245)
6	0.0126 (0.0244)	0.0124 (0.0238)	0.0038 (0.0319)	0.0126 (0.0249)	0.0032 (0.0301)	0.0089 (0.0274)	0.0173 (0.0342)	0.0074 (0.0224)
7	0.0033 (0.0348)	0.0083 (0.0319)	0.0010 (0.0291)	0.0000 (0.0198)	0.0061 (0.0344)	0.0013 (0.0148)	0.0014 (0.0308)	0.0048 (0.0262)
8	0.0381 (0.0122)	0.0213 (0.0288)	0.0348 (0.0147)	0.0266 (0.0318)	0.0338 (0.0145)	0.0150 (0.0208)	0.0332 (0.0137)	0.0200 (0.0171)
9	0.0007 (0.0311)	0.0028 (0.0205)	0.0053 (0.0339)	0.0002 (0.0255)	0.0020 (0.0326)	0.0015 (0.0165)	0.0021 (0.0274)	0.0099 (0.0150)
10	0.0008 (0.0309)	0.0057 (0.0159)	0.0024 (0.0340)	0.0060 (0.0146)	0.0010 (0.0347)	0.0099 (0.0177)	0.0025 (0.0304)	0.0063 (0.0158)

Table (4.9): The absolute bias and MSE (in brackets) of the two estimators of the SACF of the $PAR_4(2,1,0,2)$ model (Model 2), n = 50 with five additive outliers

	Season							
	1		2		3		4	
Lag	$\hat{\rho}_k$	$\breve{\rho}_k$	$\hat{\rho}_k$	$\breve{\rho}_k$	$\hat{\rho}_k$	$\breve{\rho}_k$	$\hat{\rho}_k$	$\breve{\rho}_k$
1	0.1816 (0.0571)	0.0939 (0.0311)	0.1533 (0.0480)	0.0699 (0.0285)	0.0019 (0.0208)	0.0005 (0.0150)	0.4580 (0.2381)	0.1708 (0.0725)
2	0.7377 (0.5849)	0.1246 (0.0645)	0.0345 (0.0200)	0.0183 (0.0162)	0.0009 (0.0212)	0.0044 (0.0135)	0.4838 (0.2639)	0.1817 (0.0732)
3	0.1223 (0.0370)	0.0687 (0.0247)	0.1190 (0.0352)	0.0663 (0.0228)	0.0058 (0.0245)	0.0050 (0.0122)	0.0936 (0.0276)	0.0524 (0.0193)
4	0.0010 (0.0077)	0.0060 (0.0190)	0.0010 (0.0092)	0.0069 (0.0189)	0.0201 (0.0081)	0.0125 (0.0127)	0.0303 (0.0099)	0.0219 (0.0153)
5	0.0134 (0.0273)	0.0053 (0.0115)	0.0107 (0.0252)	0.0040 (0.0140)	0.0010 (0.0188)	0.0029 (0.0124)	0.0716 (0.0255)	0.0456 (0.0133)
6	0.0168 (0.0197)	0.0036 (0.0115)	0.0012 (0.0195)	0.0010 (0.0115)	0.0106 (0.0157)	0.0118 (0.0124)	0.0055 (0.0185)	0.0008 (0.0126)
7	0.0023 (0.0215)	0.0007 (0.0121)	0.0008 (0.0181)	0.0017 (0.0128)	0.0016 (0.0207)	0.0074 (0.0110)	0.0004 (0.0213)	0.0014 (0.0109)
8	0.0223 (0.0082)	0.0189 (0.0151)	0.0189 (0.0099)	0.0102 (0.0117)	0.0204 (0.0084)	0.0162 (0.0104)	0.0200 (0.0083)	0.0117 (0.0105)
9	0.0070 (0.0155)	0.0045 (0.0095)	0.0017 (0.0182)	0.0052 (0.0113)	0.0006 (0.0210)	0.0047 (0.0080)	0.0062 (0.0196)	0.005 (0.0085)
10	0.0041 (0.0162)	0.0119 (0.0098)	0.0062 (0.0154)	0.0020 (0.0085)	0.0014 (0.0189)	0.0036 (0.0079)	0.0006 (0.0180)	0.0018 (0.0090)

Table (4.10): The absolute bias and MSE (in brackets) of the two estimators of the SACF of the $PAR_4(2,1,0,2)$ model (Model 2), n = 100 with five additive outliers

	Season							
	1		2		3		4	
Lag	$\hat{\rho}_k$	$\breve{\rho}_k$	$\hat{\rho}_k$	$\breve{\rho}_k$	$\hat{\rho}_k$	$\breve{\rho}_k$	$\hat{\rho}_k$	$\breve{\rho}_k$
1	0.1850 (0.0434)	0.0975 (0.0199)	0.1520 (0.0333)	0.0776 (0.0163)	0.0007 (0.0085)	0.0009 (0.0065)	0.4458 (0.2190)	0.1566 (0.0458)
2	0.7430 (0.5775)	0.1144 (0.0338)	0.0310 (0.0107)	0.0210 (0.0066)	0.0046 (0.0060)	0.0050 (0.0048)	0.4597 (0.2306)	0.1604 (0.0457)
3	0.1266 (0.0264)	0.0715 (0.0139)	0.1170 (0.0255)	0.0718 (0.0133)	0.0017 (0.0105)	0.0009 (0.0045)	0.0832 (0.0188)	0.0511 (0.0094)
4	0.0051 (0.0040)	0.0030 (0.0057)	0.0103 (0.0061)	0.0036 (0.0066)	0.0096 (0.0041)	0.0074 (0.0051)	0.0239 (0.0049)	0.0155 (0.0055)
5	0.0009 (0.0079)	0.0039 (0.0050)	0.0001 (0.0094)	0.0020 (0.0055)	0.0002 (0.0121)	0.0020 (0.0045)	0.0701 (0.0147)	0.0466 (0.0078)
6	0.0122 (0.0081)	0.0084 (0.0046)	0.0001 (0.0078)	0.0028 (0.0048)	0.0029 (0.0127)	0.0019 (0.0049)	0.0126 (0.0111)	0.0065 (0.0052)
7	0.0013 (0.0108)	0.0024 (0.0049)	0.0016 (0.0102)	0.0035 (0.0047)	0.0024 (0.0095)	0.0031 (0.0035)	0.0043 (0.0111)	0.0009 (0.0046)
8	0.0085 (0.0043)	0.0055 (0.0050)	0.0078 (0.0054)	0.0066 (0.0056)	0.0106 (0.0040)	0.0053 (0.0043)	0.0086 (0.0050)	0.0043 (0.0040)
9	0.0026 (0.0132)	0.0041 (0.0038)	0.0003 (0.0089)	0.0006 (0.0049)	0.0034 (0.0084)	0.0009 (0.0046)	0.0048 (0.0103)	0.0014 (0.0036)
10	0.0006 (0.0088)	0.0031 (0.0036)	0.0016 (0.0099)	0.0024 (0.0041)	0.0022 (0.0091)	0.0037 (0.0042)	0.0035 (0.0092)	0.0056 (0.0043)

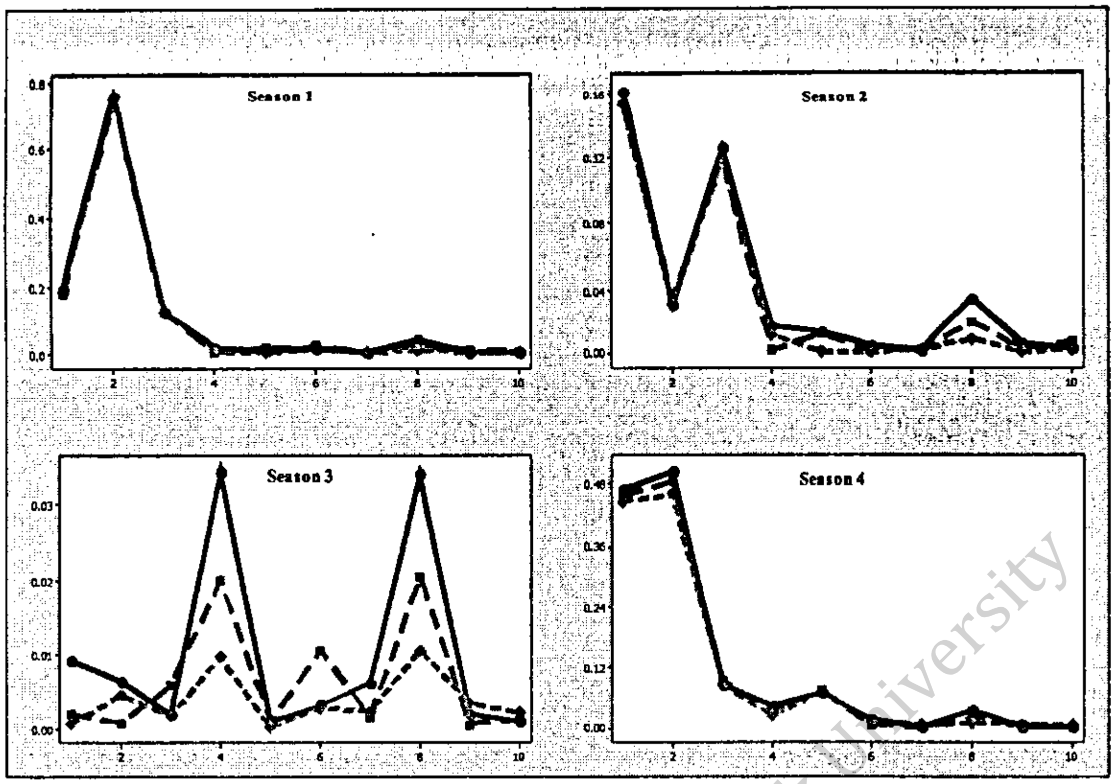

Figure (4.9): The absolute bias for $\hat{\rho}_k(\nu)$ for $PAR_4(2,1,0,2)$ model (Model 2) with five additive outliers (——: n=30, — — : n=50, ----: n=100)

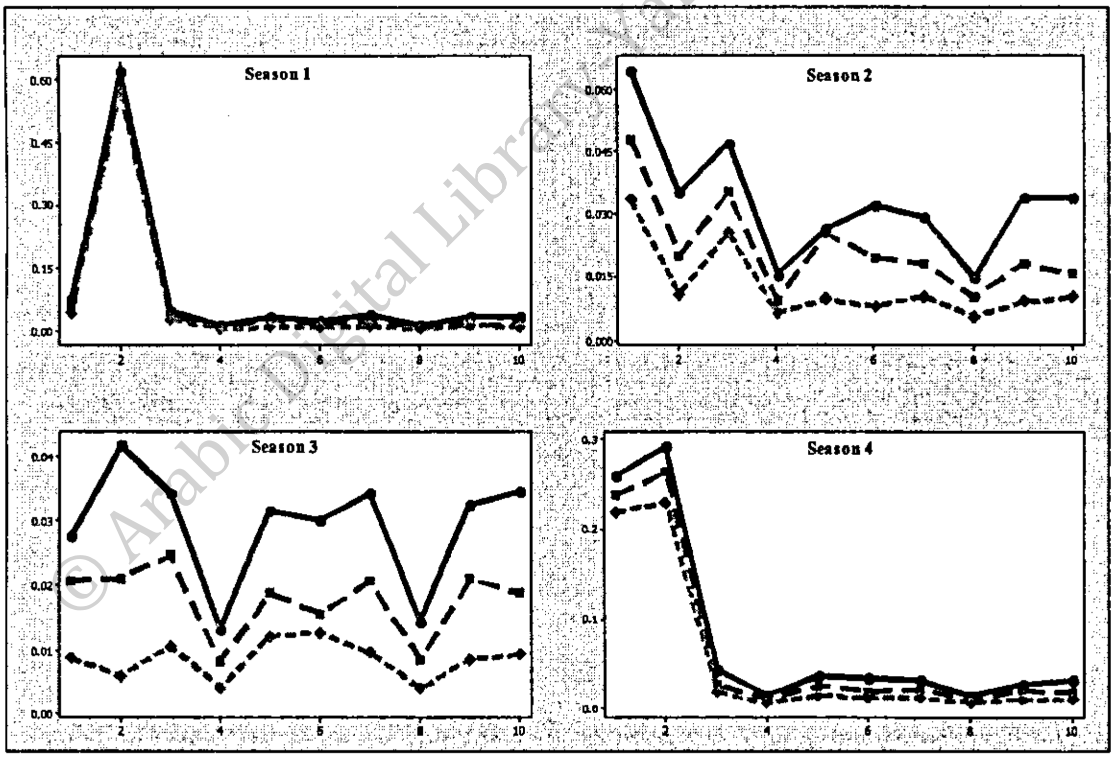

Figure (4.10): The MSE for $\hat{\rho}_k(\nu)$ for $PAR_4(2,1,0,2)$ model (Model 2) with five additive outliers (——: n=30, — — : n=50, ----: n=100)

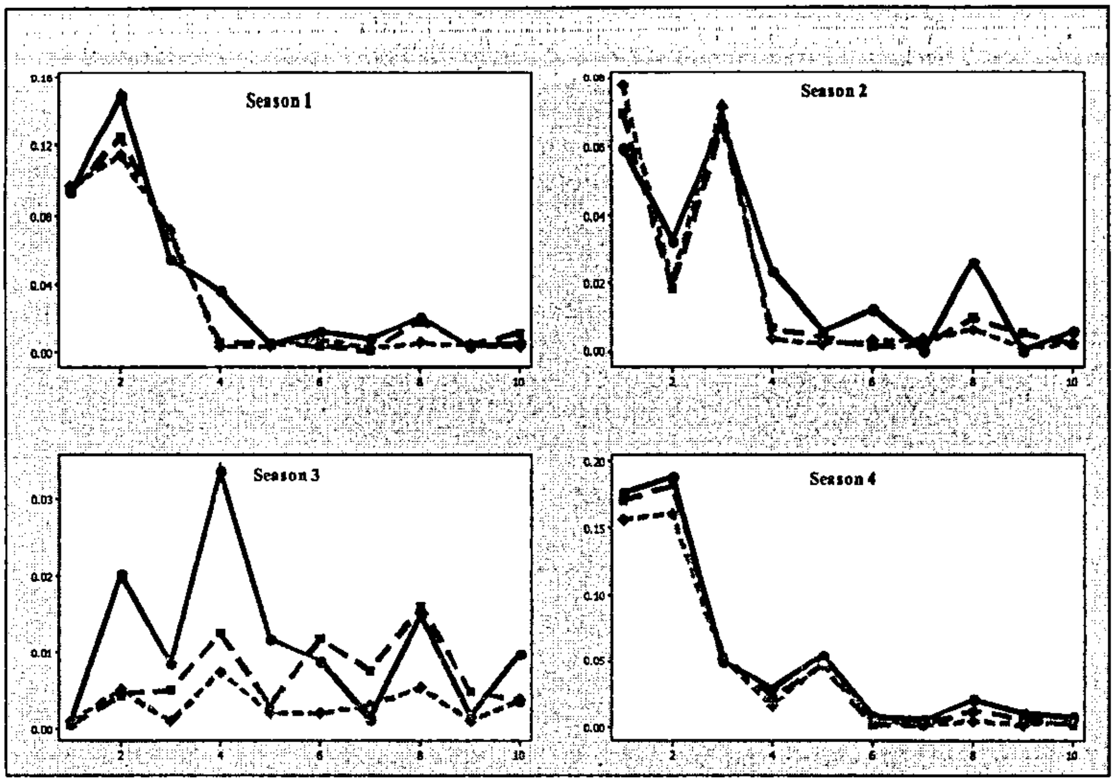

Figure (4.11): The absolute bias for $\breve{\rho}_k(\nu)$ for $PAR_4(2,1,0,2)$ model (Model 2) with five additive outliers (—: n=30, – – : n=50, ----: n=100)

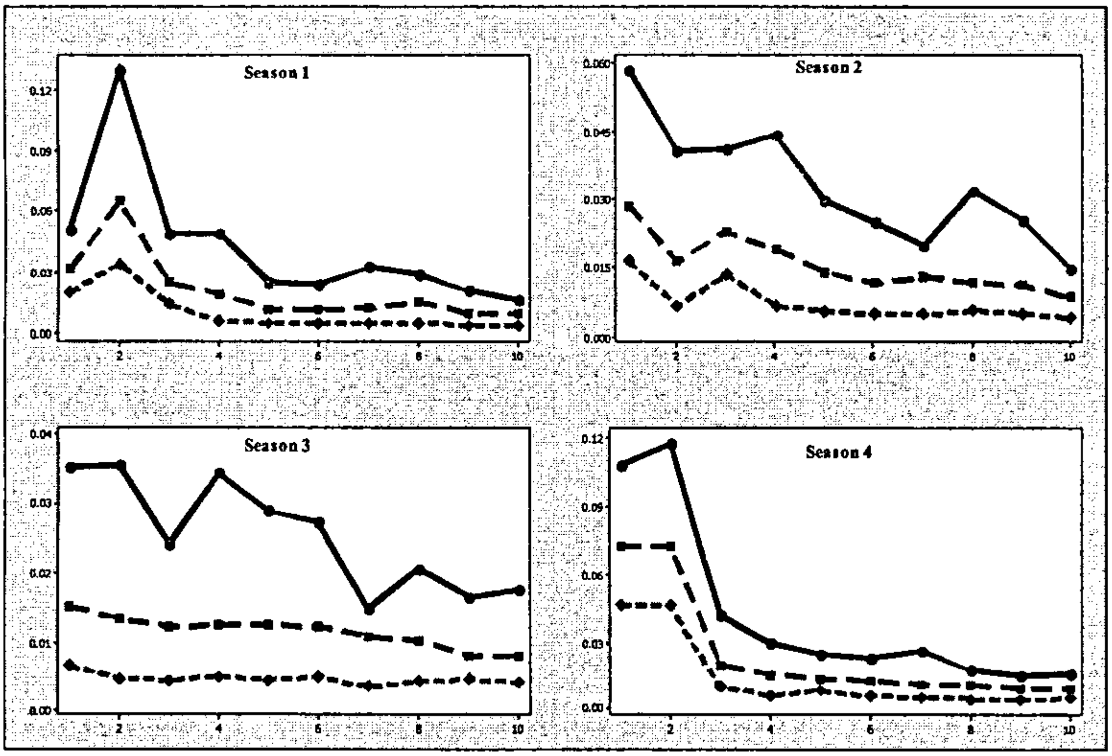

Figure (4.12): The MSE for $\breve{\rho}_k(\nu)$ for $PAR_4(2,1,0,2)$ model (Model 2) with five additive outliers (—: n=30, – – : n=50, ----: n=100)

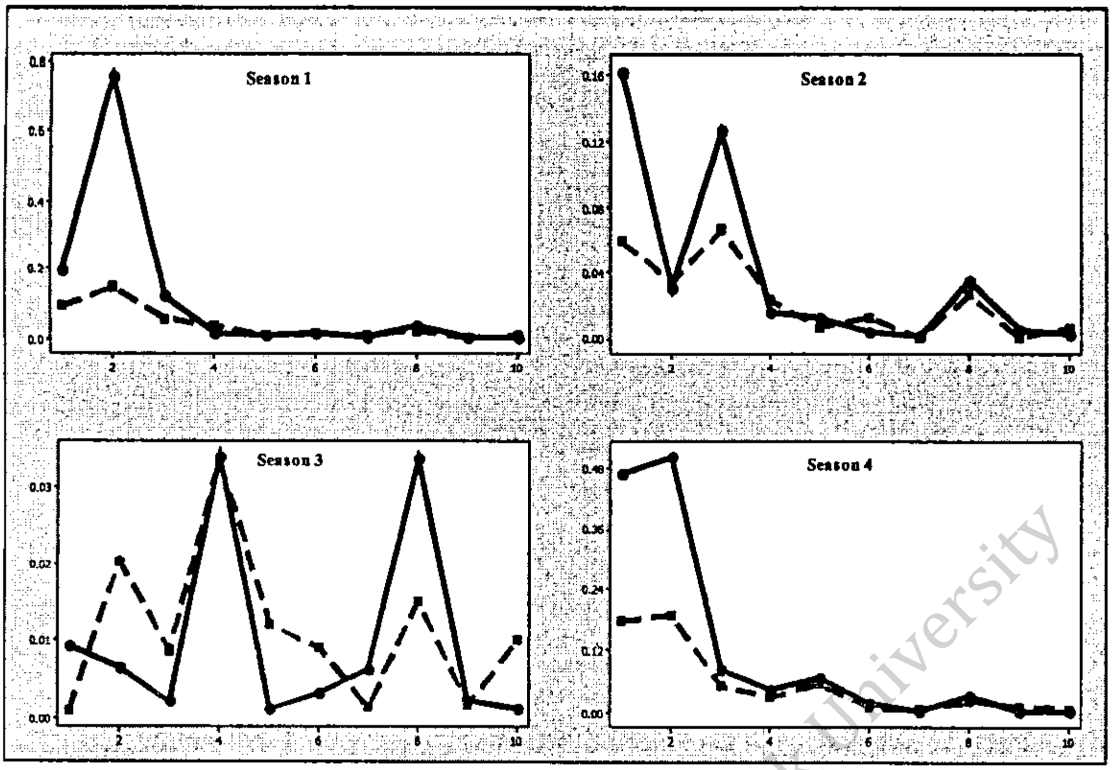

Figure (4.13): The absolute bias for the two estimators for $PAR_4(2,1,0,2)$ model (Model 2) with five additive outliers and n=30 (—: $\hat{\rho}_k(\nu)$, – – : $\breve{\rho}_k(\nu)$)

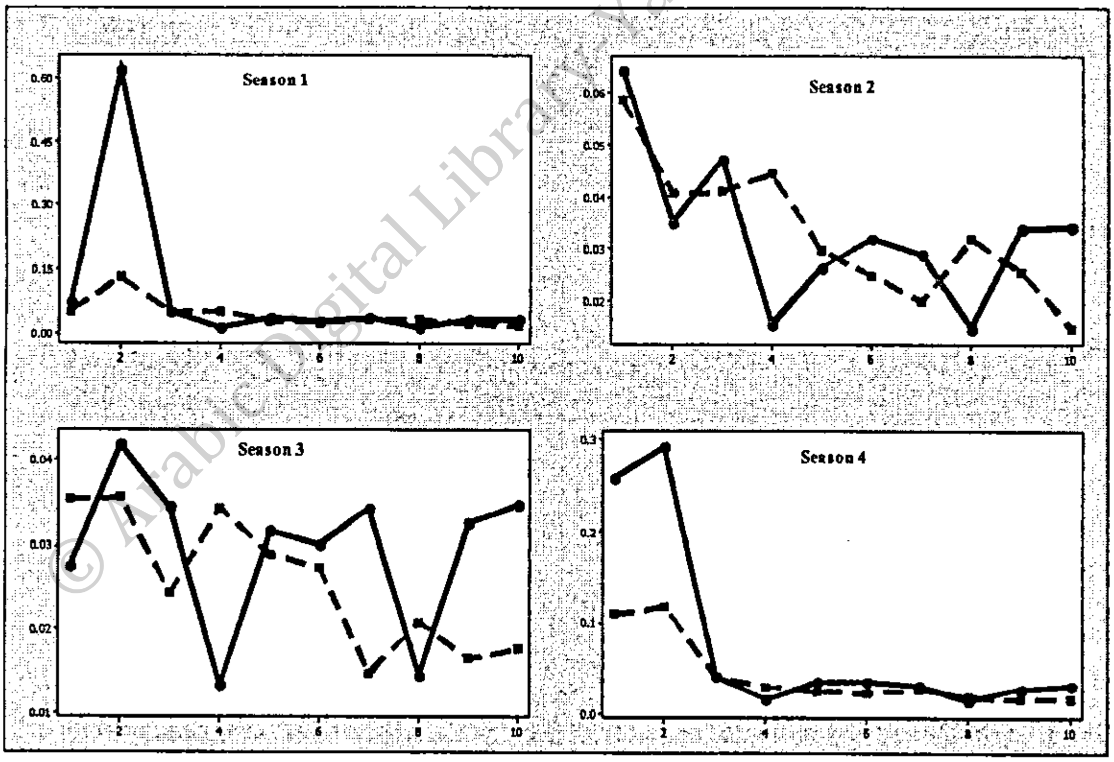

Figure (4.14): The MSE for the two estimators for $PAR_4(2,1,0,2)$ model (Model 2) with five additive outliers and n=30 (—: $\hat{\rho}_k(\nu)$, – – : $\breve{\rho}_k(\nu)$)

4.6 Results

From Tables (4.5)-(4.7) and Figures (4.3)-(4.6) we can notice that the two estimators of Model 1 seem to be asymptotically unbiased since the absolute bias decreases as n increases. Also, both estimators seem to be consistent sense the MSE goes to zero as n increases. Similar results apply for the varying orders PAR model; Model 2 as can be seen from Tables (4.8)-(4.10) and Figures (4.9)-(4.12).

In the case of a single additive outlier of Model 1, the moment estimator $\hat{\rho}_k(\nu)$ has a larger absolute bias and MSE in season one in which the outlier was presented, this fact was true for all n. So, we can say that the estimator $\breve{\rho}_k(\nu)$ of Model 1 was more efficient than the moment estimator $\hat{\rho}_k(\nu)$ (see Tables (4.5)-(4.7) and Figures (4.7) and (4.8)).

In the case of five random additive outliers of Model 2, we can see that the second estimator $\breve{\rho}_k(\nu)$ has smaller absolute bias and MSE than the moment estimator $\hat{\rho}_k(\nu)$ at seasons one and four. While the bias and MSE are close to each other for the both estimators at seasons two and three. So, we may say that the second estimator $\breve{\rho}_k(\nu)$ seems to be a robust estimator for $\rho_k(\nu)$.

For $\hat{\rho}_k(\nu)$ of Model 1 the absolute bias and MSE decreases as time lag increases at season one which contains the outlier and this is not clear for the other seasons which don't contain any outliers. While for $\breve{\rho}_k(\nu)$ the absolute bias and MSE seems to be not decreases as time lag increases. For large lags the absolute bias and MSE values became very close especially at season one.

Note that our results are true, namely for the selected models and cases.

CHAPTER 5

Application to Real Data

5.1 Introduction

In this chapter we are going to apply our work to real time series data. It is known that one of the first applications of PAR models was for hydrological time series (Hiple and Mclead, 1994).

Thomas and Fiering (1962) originally suggested that one could fit PAR (1) models to monthly (Quarterly) hydrological time series and since our interest is to check the robustness of our estimators of $\rho_k(\nu)$ not to find the best model of the data we will assume our data follows a PAR(1) model. So, our aim is to apply the estimators (3.4), (3.5) and (3.6) for the SACF of PAR(1) model and compare them via their standard errors (S.E) for some real data.

5.2 The Data

We chose to apply our methods on a hydrological monthly time series, namely the monthly river flows of Fraser river at hope. According to Wikipedia:"Fraser River is the longest river within British Columbia, Canada, rising at Fraser Pass near Mount Robson in the Rocky Mountains and flowing for 1,375 km, into the Strait of Georgia at the city of Vancouver. It is the tenth longest river in Canada. The river's volume at its mouth is 112 km³ each year (about 800,000 gal/s or 3550 cubic meters per second)".

The data expand for 78 years from January 1914 to December 1991 (Hiple and Mcleod, 1994). For simplicity, we transformed (aggregated) this time series into quarterly time series such that:

- January, February and March stands for quarter one (Q1).
- April, May and June stands for quarter two (Q2).
- July, August and September stands for quarter three (Q3).
- October, November and December stands for quarter four (Q4).

The time series plot of our quarterly time series is given in Figure (5.1) and a parallel box-plot of marginal data for each quarter is shown in Figure (5.2). Figure (5.1) shows an apparent (constant) seasonality with nearly no-trend.

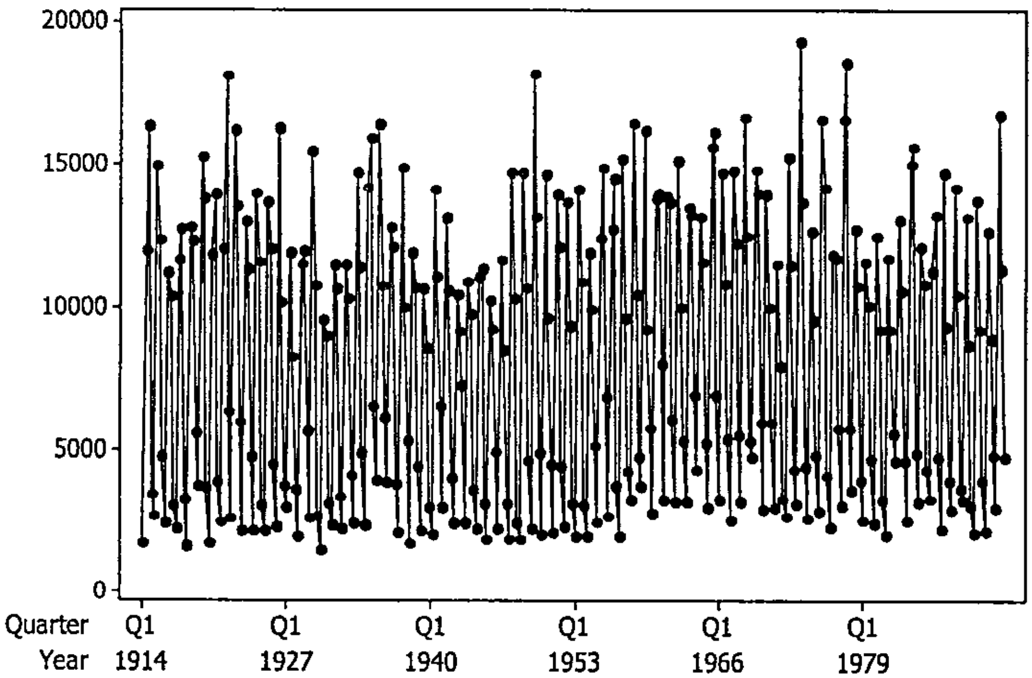

Figure (5.1): The time series plot of quarterly sums data

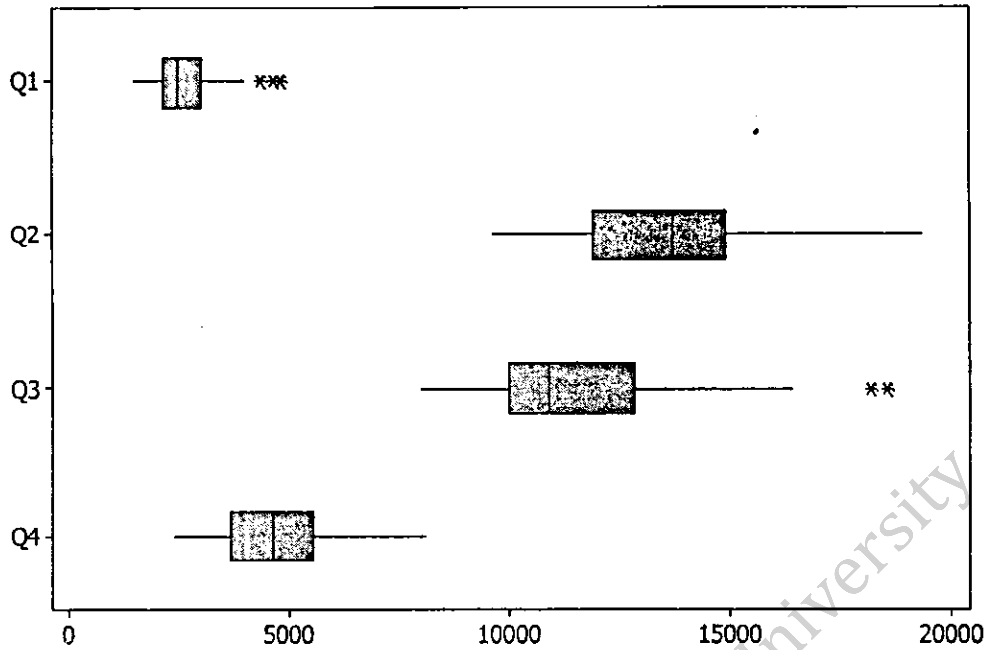

Figure (5.2): The box plot of quarterly (marginal) data

From Figure (5.2) we can see that we have some outliers, three at season one (Q1) and the other two at season three (Q3). This approach for detecting outliers in such time series was done for example by Shoa (2007). Also note that the medians for various seasons are different with the largest at quarter two and the smallest at quarter one. This means that the largest river-flow is at April, May and June while the smallest river-flow is at January, February and March. Besides, the variability was the largest in Q2 and smallest in Q1.

5.3 Methodology and Analysis

Assuming that the quarterly time series of previous section comes from the $PAR_4(1)$ model, then our interest is to estimate the SACF of this process. As our data is a single realization then to be able to compare various estimators via their standard error we used bootstrap method.

Besides, in chapter three we have computed the bias of each estimator as the theoretical PAR model along its parameters were known. Here, we will only give the average estimators of SACF based on (3.4)-(3.6).

Therefore, we have selected 20 random samples each of length 10 (consecutive) years. Then, the average estimates of SACF and its standard error are given in terms of $r_k(\nu)$ as:

$$\bar{r}_k(\nu) = \frac{1}{20}\sum_{j=1}^{20}(r_k(\nu))_j$$

and

$$S.E = \sqrt{\frac{1}{20}\sum_{j=1}^{20}\left((r_k(\nu))_j - \bar{r}_k(\nu)\right)^2}\ .$$

These formulas are used for the three estimators (3.4)-(3.6).

We have done the computations using R-package. The results are summarized in Table (5.1) and Figure (5.3).

Table (5.1): The averages of the three estimators of the SACF and their S.E (in brackets)

	Season											
	1			2			3			4		
Lag	$\hat{\rho}_k$	$\tilde{\rho}_k$	$\breve{\rho}_k$	$\hat{\rho}_k$	$\tilde{\rho}_k$	$\breve{\rho}_k$	$\hat{\rho}_k$	$\tilde{\rho}_k$	$\breve{\rho}_k$	$\hat{\rho}_k$	$\tilde{\rho}_k$	$\breve{\rho}_k$
1	0.4501 (0.2394)	0.2776 (0.3276)	0.2550 (0.2420)	0.1364 (0.3199)	-0.0030 (0.4269)	0.0432 (0.3200)	0.2468 (0.3603)	0.2131 (0.5931)	0.1370 (0.4030)	0.4773 (0.2356)	0.3747 (0.4396)	0.2730 (0.3923)
2	0.1396 (0.2077)	0.0678 (0.1313)	0.0786 (0.1630)	0.1127 (0.4279)	0.0195 (0.1746)	0.0102 (0.1280)	0.1185 (0.2488)	-0.0206 (0.3058)	0.0016 (0.1510)	-0.0305 (0.2278)	0.0569 (0.3033)	-0.0384 (0.1931)
3	0.0758 (0.2754)	0.0377 (0.0996)	-0.0195 (0.0796)	0.1559 (0.2965)	0.0191 (0.0610)	0.0025 (0.0279)	0.0218 (0.3471)	0.0203 (0.1479)	0.0073 (0.0488)	-0.0372 (0.2209)	0.0302 (0.1124)	0.0131 (0.0208)
4	-0.1700 (0.2944)	0.0150 (0.0434)	0.0041 (0.0078)	0.0269 (0.2438)	0.0150 (0.0434)	0.0041 (0.0078)	-0.0602 (0.2570)	0.0150 (0.0434)	0.0041 (0.0078)	-0.1148 (0.2732)	0.0150 (0.0434)	0.0041 (0.0078)
5	-0.2014 (0.2354)	0.0094 (0.0271)	0.0017 (0.0037)	-0.0718 (0.2786)	0.0069 (0.0252)	0.0007 (0.0040)	-0.1422 (0.2757)	0.0125 (0.0331)	0.0008 (0.0042)	0.0036 (0.4222)	0.0083 (0.0289)	0.0019 (0.0042)
6	-0.0886 (0.2894)	0.0041 (0.0127)	0.0008 (0.0018)	-0.1020 (0.2579)	0.0051 (0.0136)	0.0004 (0.0018)	0.0825 (0.2712)	0.0078 (0.0173)	0.0002 (0.0018)	0.0053 (0.2361)	0.0056 (0.0219)	-0.0004 (0.0026)
7	0.0242 (0.1751)	0.0029 (0.0092)	-0.0001 (0.0011)	-0.0624 (0.2365)	0.0026 (0.0072)	0.0001 (0.0007)	-0.0119 (0.3296)	0.0046 (0.0103)	0.0001 (0.0007)	0.1954 (0.2971)	0.0048 (0.0113)	0.0002 (0.0005)
8	-0.0591 (0.1733)	0.0021 (0.0050)	0.0001 (0.0002)	-0.1325 (0.2540)	0.0021 (0.005)	0.0001 (0.0002)	-0.0293 (0.2033)	0.0021 (0.0050)	0.0001 (0.0002)	-0.0406 (0.2743)	0.0021 (0.0050)	0.0001 (0.0002)
9	-0.0536 (0.1759)	0.0012 (0.0027)	0.0000 (0.0001)	-0.1219 (0.2564)	0.0011 (0.0030)	0.0000 (0.0001)	0.0081 (0.2671)	0.0014 (0.0036)	0.0000 (0.0001)	0.0422 (0.1755)	0.0012 (0.0036)	0.0000 (0.0001)
10	-0.0146 (0.2302)	0.0006 (0.0016)	0.0000 (0.0000)	0.0045 (0.2561)	0.0006 (0.0016)	0.0000 (0.0001)	-0.0385 (0.3209)	0.0007 (0.0021)	0.0000 (0.0000)	0.0531 (0.2480)	0.0009 (0.0027)	0.0000 (0.0000)

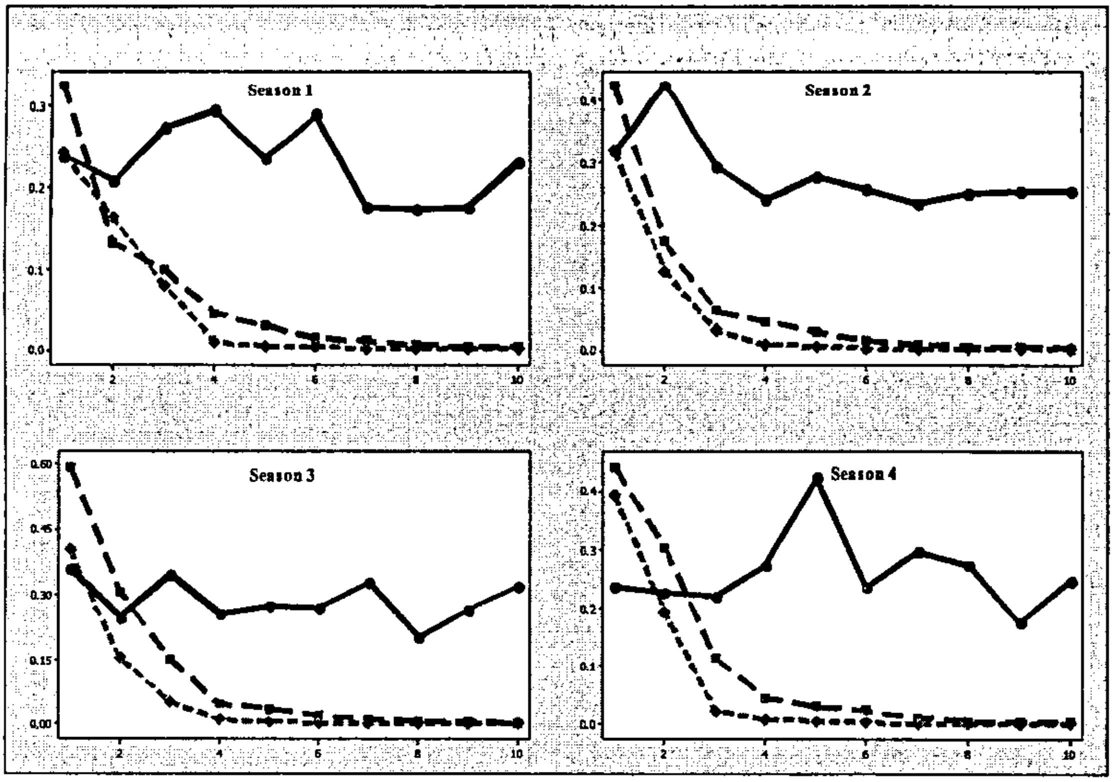

Figure (5.3): The S.E for the three estimators of the real data for $\text{PAR}_4(1)$ model
(——: $\hat{\rho}_k(\nu)$, — — : $\tilde{\rho}_k(\nu)$, ----: $\breve{\rho}_k(\nu)$)

We can see from Table (5.1) and Figure (5.3) that the moment estimator $\hat{\rho}_k(\nu)$ has S.E value higher than the other two estimator $\tilde{\rho}_k(\nu)$ and $\breve{\rho}_k(\nu)$. This means that $\tilde{\rho}_k(\nu)$ and $\breve{\rho}_k(\nu)$ seem to resist the existence of the outliers while $\hat{\rho}_k(\nu)$ does not. This result coincides with what we have found in chapter three.

Finally, we recall that the robust estimator given in (3.5) is based on the fact that the original model is PAR(1) model while the moment estimator (3.4) and the other robust estimator (3.6) are model-free. That is, the results in this chapter are still valid for $\hat{\rho}_k(\nu)$ and $\breve{\rho}_k(\nu)$ without assuming any specific PAR model.

CHAPTER 6

Conclusions and Future Work

6.1 Introduction

In this chapter we are going to summarize the most important results in this thesis. Besides, we will give some ideas for possible future research.

6.2 Conclusions

In this thesis we have investigated robust estimation for the SACF for various PAR models. We have found many useful results concerning the PAR(1), PAR(2) and varying-order PAR models.

For the PAR(1) model we investigated three estimators for SACF including the moment estimator beside two other estimators.

For the computation of bias and MSE of various estimates of SACF we have developed some theoretical formulas to compute the theoretical SACF of various PAR models. Besides, the theoretical SACF of some PAR models are computed and sketched. According to the simulation results, we found that the moment estimator was affected by the existence of additive outliers in view of bias and MSE criterion while the other estimators seem to be more robust estimators and this is true for different periods and different number of additive outliers at different seasons. In particular, the moment estimator was highly affected in the season(s) that contained additive outlier(s).

For the PAR(2) and varying-orders PAR(2,1,0,2) models we also investigate two estimators for SACF including the moment estimator. We cover two cases of adding outliers, the first case is by adding a single additive outliers at season one and the second case is by adding several random (in view of magnitudes and positions) outliers.

Again, it is noticed that the moment estimator was highly affected by outliers than the other (robust) estimators.

Finally, we apply our work on real time series data of river flow. We use the real data to compute the three estimators for SACF that we used for PAR(1) model then, we found that the moment estimator was highly affected by the existence of additive outliers while the other estimators seem to be more robust estimators.

6.3 Future Work

From a research point of view, time series analysis and modeling is a very important area for both theoretical and applied disciplines. Although many publications are available on the PAR and PARMA models, we believe that there are several issues regarding these models that need further researches. In view of our results we summarize some possible ideas for future work as follows:

- Investigating robust estimation for the SACF for other models like PMA and PARMA models for different time lags and periods.
- Developing other robust estimators using trimmed mean and MAD.

- Studying the robustness of the estimators in the existence of innovative outliers.
- Studying the robustness of moment estimator regarding issues other than outliers, as for instance the shape of distribution of the random errors.

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